AN INTRODUCTION

TO THE

LOGIC OF ALGEBRA.

With Illustrative Exercises.

BY

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NEW YORK:
JOHN WILEY & SONS,
53 EAST TENTH STREET.
1890.

ROBERT DRUMMOND,
Electrotyper,
444 & 446 Pearl Street,
New York.

FERRIS BROS.,
Printers,
326 Pearl Street,
New York.

ERRATA.

Page viii, line 30. *For* $a - (\pm b)a = \mp b$ *read* $a - (\pm b) = a \mp b$

" *xi, line* 3. *For* th$ *read* the

" 3, " 11. " $a + b$ *read* $b + a$

" 7, " 8. " a " a^b

" 24, " 3. " $(- b$ " $(- b)$

" 29, " 40. " 55 " 56

" 38, " 14. " of positive *read* of a positive

" 40, " 17. " fraction " fractions

" 49, *diagram.* " o " O

" 52, *line* 12. " thou sands *read* thousands

" " " " " thm " them

" 52, " 14. " $- a^1$ " $- a^{\frac{1}{2}}$

" 61, " 9. " $a_1 = \dfrac{y - y_1}{x - x_1}$ *read* $a, = \dfrac{y - y_1}{x - x_1}$,

" 62, *diagram. Leftmost sloping line should join* — 3h *and* 1

" 69, *lines* 8 *and* 16. *For* integrably *read* integrally

" 73, *line* 6. *For* $- 2 - 2k$ *read* $= 2 - 2k$

" 78, " 20. " $11 > 0$ " $11 \ngtr 0$

" 85, *diagram. The dot for* $\overline{17}$ *is out of place*

" 87, *line* 26. *For* arrows *read* numbers

" 98, " 17. " q^1 " q'

" 102, " 12. " $T(1 + i^{\frac{1}{1024}}$ *read* $T(i + i^{\frac{1}{1024}})$

" 106, " 23. " (eki^l) " $(e^k i^l)$

" 111, *diagram.* F *wanted at end of sloping line from* A

" 112, *line* 20. *For* $e^{1+(2n+1)i}$ *read* $e^{1+(2n\pi+1)i}$

" 112, " 30. " i' *read* i

" 113, " 18. " i' " i

" 113, " 32. " $ib' - b$ *read* $i(b' - b)$

" 114, " 23. " r_s " r_b

" 115, " 4. " $1_{\frac{\pi}{2}}$ " $1_{-\frac{\pi}{2}}$

DEDICATION.

TO MY DECEASED FATHER,

The Reverend L. W. Davis,

IN GRATEFUL RECOGNITION OF HIS CAREFUL TRAINING,

HIS INEXHAUSTIBLE PATIENCE,

AND HIS SELF-SACRIFICING EFFORTS TO FURTHER MY EDUCATION.

PREFACE.

THIS book is precisely described by the title, and is mainly the outgrowth of a conviction that the *logic* of algebra is a very much neglected study.

Partly because the processes of algebra are simple and easily taught, partly because both arithmetic and algebra are generally studied for the sake of the processes rather than for the sake of discipline, the reasoning which underlies the processes has come to be very generally slurred over or even absolutely ignored.

This fault would be somewhat overcome if geometry were taught before or along with algebra, so that geometric illustrations could constantly be given of algebraic principles. The conceptions of geometry are less abstract and so more easy to grasp than those of algebra, and the reasoning of geometry is correspondingly more simple. Furthermore, geometry is usually presented as a fixed and settled science received in all its perfection from the hands of the immortal Euclid. The student does not learn one definition of a triangle and then unlearn it for another: is not told that straightness is only a special case of crookedness; that the inside of a circle is only the outside looked at in a peculiar way.

Algebra, on the other hand, shows everywhere traces of its origin and development. Numbers are, first, integral and positive; afterwards, negative, fractional, incommensurable, imaginary, and double. There is a corresponding series of meanings to the words sum, difference, product, quotient, power, root, and logarithm. Moreover, these extensions of meaning are all more or less arbitrary, and some of them at first sight contradictory. One has constantly to discriminate between

just what is argument and just what is definition and assumption on which future argument is to be based.

But these very difficulties make the logic of algebra a peculiarly useful and invigorating discipline. Besides, there is no branch of mathematics which gives one so good a general insight into the whole body of mathematics. By it the student learns the meaning and relationship of processes that he has been using more or less blindly, perceives the oneness of mathematical reasoning whether veiled under the name of geometry or of algebra, and gets a glimpse of those methods and conceptions on which the whole of modern mathematics has been built up.

In the hope that the present little book may contribute to this desirable end it is submitted to the indulgence of teachers. Only an introduction is attempted, because an attempt at more would defeat the very end in view. Thorough discussion of a few things better trains the mind than a superficial treatment of many. My only fear is lest I shall have erred in giving too much.

The student is supposed to have a knowledge of geometry and elementary algebra. In Part Second, some knowledge of trigonometry and analytic geometry will be a help.

In the preparation of the book various sources have been drawn upon. These are the more important:

ARGAND, Sur le manière de représenter quantités imaginaires.
CLIFFORD, Common Sense of the Exact Sciences.
DE MORGAN, Trigonometry and Double Algebra.
" Calculus, Introduction.
DIRICHLET, Zahlen Theorie.
TANNERY, Théorie des Fonctions d'une variable seule.

My thanks are particularly due to Prof. Sloan of the University of South Carolina for help in revising manuscript, and to Mr. Gustav Bissing of Baltimore for many corrections and fruitful suggestions.

UNIVERSITY OF SOUTH CAROLINA,
January 1, 1890.

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PART FIRST.

SIMPLE NUMBERS.

I. The Direct Processes with Positive Integers.

1. Mathematics is characterized by deductive reasoning.
2. Reasoning requires language. By growth of language Arithmetic becomes Algebra.
3. The number of objects in a group is independent of the order of counting.
4. Addition is commutative: $a+b=b+a$.
5. Addition is associative: $(a+b)+c=a+(b+c)$.
6. Multiplication is commutative: $a\times b=b\times a$.
7. Multiplication is associative for three numbers: $(a\times b)\times c=a\times(b\times c)$.
8. The product of any number of factors is independent of the order in which they are multiplied together.
9. Multiplication is associative for any number of factors.
10. Involution grows out of multiplication as does multiplication out of addition.
11. Involution is non-commutative: $a^b \neq b^a$.
12. Involution is non-associative: $a^{(b^c)} \neq (a^b)^c$.
13. The direct processes are uniform: $a=a'$ with $b=b'$ requires $a+b=a'+b'$, $a\times b=a'\times b'$, and $a^b=a'^{b'}$.
14. Multiplication takes precedence of involution; addition, of both multiplication and involution: $a+b\times c^d=a+\left[b\times(c^d)\right]$, a convention.
15. Multiplication is distributive to addition: $(b+c)\times a=b\times a+c\times a$.
16. Addition is non-distributive to multiplication: $(b\times c)+a\neq(b+a)\times(c+a)$.
17. Involution is distributive to multiplication when the product is the base, but not when the product is the index: $(b\times c)^a=b^a\times c^a$, but $a^b\times a^c\neq a^{b\times c}$.
18. Multiplication is non-distributive to involution: $b^c\times a\neq(b\times a)^{c\times a}$.
19. Involution is non-distributive to addition: $(b+c)^a\neq b^a+c^a$, $a^{b+c}\neq a^b+a^c$.
20. Addition is non-distributive with regard to involution; $b^c+a\neq(b+a)^{c+a}$.
21. The distributive law for involution is $(b\times c)^a=b^a\times c^a$. The index laws are $a^b\times a^c=a^{b+c}$ and $(a^b)^c=a^{b\times c}$.
22. Successive powerings can be performed in any order: $(a^b)^c=(a^c)^b$.

II. The Inverse Processes with Positive Integers.

23. The inverse processes are the undoers of the direct processes.
24. Subtraction undoes addition: if $a+b=c$, $c-b=a$.
25. Division undoes multiplication: if $a\times b=c$, $c\div b=a$.
26. Involution may be undone in two ways.
27. Evolution determines an unknown base: if $a^b=c$, $a=\sqrt[b]{c}$.
28. Taking a logarithm determines an unknown index: if $a^b=c$, $b=\log_a c$.
29. A whole series of processes may be undone: if $(\overline{a+b}\times c)^d=e$, $a=\sqrt[d]{e}\div c-b$.
30. The inverse processes are performed by 'guess and try.'
31. The inverse processes are non-commutative: $a-b\neq b-a$, $a\div b\neq b\div a$, $\sqrt[b]{a}\neq\sqrt[a]{b}$, $\log_a b\neq\log_b a$.
32. The inverse processes are non-associative: $(a-b)-c\neq a-(b-c)$, $(a\div b)\div c\neq a\div(b\div c)$, $\sqrt[a]{\sqrt[b]{c}}\neq\sqrt[\sqrt{b}]{c}$, $\log_a\log_b c\neq\log_{\log_a b} c$.
33. Parentheses are sometimes avoided by conventions as to the use of the symbols $+$, $-$, $\times$, $\div$, $(\ \ )^{(\ )}$, $\sqrt{}$, log.

III. Negatives and Fractions.

34. The inverse processes lead to new numbers: $a-b\overset{>}{\underset{<}{=}}c-d$ if $a+d\overset{>}{\underset{<}{=}}b+c$.
35. To every positive number, $+a$, corresponds a negative number, $-a$.
36. A positive number is a name reached by counting on forward from zero; a negative, one reached by counting off backward from zero.
37. Multiplying a $\left\{\begin{array}{c}\text{dividend multiplies}\\ \text{divisor divides}\end{array}\right\}$ the quotient.
38. $a\div b\overset{>}{\underset{<}{=}}c\div d$ if $a\times d\overset{>}{\underset{<}{=}}b\times c$.
39. $a\div b=\frac{a}{b}$, a fraction.
40. The bar of a fraction is a sign of inclusion.
41. $3\times 5\times a\times b\times(c+d)=3.5ab(c+d)$. Rules of precedence.
42. Between fractions lie other fractions forever: if $\frac{a}{b}>\frac{c}{d}$, $\frac{a}{b}>\frac{a+c}{b+d}>\frac{c}{d}$.
43. The sum of two integers is the integer reached by counting from either as we would count from zero to get the other.
44. Addition is indicated by writing numbers together connected by their proper signs: $(+a)+(-b)=a-b$. To subtract a number is to add its opposite: $a-(\pm b)a=\mp b$.
45. The order of algebraic addition is indifferent: $+a-b=-b+a$.
46. Algebraic addition is both commutative and associative.
47. To subtract an addition and subtraction expression is to add its opposite.

48. In an involved addition and subtraction expression the sign of any number is, upon the whole, $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right\}$ if an $\left\{\begin{matrix}\text{even}\\ \text{odd}\end{matrix}\right\}$ number of minus signs act upon it.

49. Addition and subtraction of integers gives naught save integers.

50. The product of two numbers is $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right\}$ if they have $\left\{\begin{matrix}\text{the same sign}\\ \text{opposite signs}\end{matrix}\right\}$.

51. The product of any number of factors n of which are negative is $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right\}$ if n is $\left\{\begin{matrix}\text{even}\\ \text{odd}\end{matrix}\right\}$.

52. Multiplication, when negatives enter, continues to be commutative and associative.

53. The quotient of two numbers is $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right\}$ if they have $\left\{\begin{matrix}\text{the same sign}\\ \text{opposite signs}\end{matrix}\right\}$.

54. Between any two negative fractions lie an infinite number of other negative fractions.

55. To multiply by a fraction means to multiply by the numerator and then divide by the denominator.

56. The product of any number of fractions is the fraction $\dfrac{\text{product of numerators}}{\text{product of denominators}}$. The operation is commutative and associative.

57. To $\left\{\begin{matrix}\text{multiply}\\ \text{divide}\end{matrix}\right\}$ by a number is to $\left\{\begin{matrix}\text{divide}\\ \text{multiply}\end{matrix}\right\}$ by its reciprocal.

58. The result of a chain of multiplications and divisions is independent of the order in which they are performed.

59. Any number in an involved multiplication and division expression is, upon the whole, a $\left\{\begin{matrix}\text{multiplier}\\ \text{divisor}\end{matrix}\right\}$ of that expression if it is acted upon by an $\left\{\begin{matrix}\text{even}\\ \text{odd}\end{matrix}\right\}$ number of division signs.

60. We cannot by multiplication and division with positive integers and fractions get aught save positive integers and fractions.

61. The sum of any number of fractions is a new fraction, whose numerator is the sum of all the products obtained by multiplying the numerator of each given fraction by the denominators of all the other given fractions, and whose denominator is the product of all the given denominators.

62. Subtraction of fractions enters as did subtraction of integers, carrying with it negative fractions and algebraic addition and subtraction of fractions.

63. §§ 50, 51, 52, 53, hold for fractions. To multiply by a multiplication and division expression is to divide by its reciprocal.

64. With the numbers now introduced, addition, subtraction, multiplication, and division are always possible.

65. A multiplication and division expression is powered by distributing the index of the power over the factors of the expression.

66. $\left(\dfrac{a}{b}\right)^c = \dfrac{d}{e}$ requires $\sqrt[c]{\dfrac{d}{e}} = \dfrac{a}{b}$ and $\log_{\frac{a}{b}} \dfrac{d}{e} = c$.

67. Powering by a $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right\}$ integer is repeatedly $\left\{\begin{matrix}\text{multiplying}\\ \text{dividing}\end{matrix}\right\}$ unity by the base. Powering by zero is letting unity alone.
68. A fractional power is an integral root of an integral power.
69. $a^{\frac{b}{c}} = \sqrt[\frac{c}{b}]{a}$. If $\left(\frac{a}{b}\right)^{\frac{c}{d}} = \frac{e}{f}$, $\log_{\frac{a}{b}} \frac{e}{f} = \frac{c}{d}$.

IV. Incommensurables.

70. Evolution leads to new expression numbers: $\sqrt{2}$ is no integer or fraction.
71. $\sqrt[b]{\frac{a}{e}} \gtreqless \sqrt[d]{\frac{c}{f}}$, if $\left(\frac{a}{e}\right)^{d} \gtreqless \left(\frac{c}{f}\right)^{b}$.
72. Opposites and reciprocals of incommensurables lie between the opposites and reciprocals, respectively, of the inclusives of the incommensurables.
73. Taking logarithms leads to incommensurables: $\log_2 3$ is no integer or fraction.
74. If $\left(\frac{a}{b}\right)^{\frac{c}{d}} > \frac{e}{f} > \left(\frac{a}{b}\right)^{\frac{g}{h}}$, then $\log_{\frac{a}{b}} \frac{e}{f}$ lies between $\frac{c}{d}$ and $\frac{g}{h}$.
75. An incommensurable is a number dividing all fractions into two sets A and B, so that any fraction from A is less than any from B, but yet no fraction from A is largest, nor any from B smallest.
76. Two incommensurables are equal if their inclusives are equal.
77. The results of operating with incommensurables are hemmed in by the results of operating with their inclusives.
78. As always, the inverse operations are mere undoers of the direct.
79. To hem in all incommensurables, we do not need all fractions: decimal fractions are sufficient.
80. Ratio is a general term for all sorts of numbers. The sign ' : ' is not identical with the sign '÷'.

V. Illustrations.

81. Algebraic numbers and their addition and subtraction are illustrated by steps and otherwise. Negative numbers are sometimes nonsense.
82. The multiplication of algebraic numbers is illustrated by a lever.
83. The illustrations of §§ 81, 82, may be extended to fractions, but fractions are sometimes non-sense.
84. Lengths taken at random are very likely incommensurable. The illustration of the lever is extended to incommensurables.

VI. Growth and Rate.

85. The foregoing sections comprise the main conceptions of elementary algebra.
86. A number grows from one value to another by taking in succession all intermediate values.

87. When $y = ax$, y grows with x at a uniform rate a.

88. When $y = x^2$, y grows at varying rate $2x$ compared to x. The ratio $\frac{y' - y'}{x' - x'}$ has a definite value only because of the law connecting y and x in the growth of $\frac{y' - y'}{x' - x'}$ from $\frac{y - y'}{x - x'}$.

89. Whatever law connects y growth with x growth, the varying rate of growth of y compared to x is $\frac{y - y}{x - x}$.

90. If $y = a^x$, the rate of y growth to x growth when $x' = 0$ is hemmed in by $\frac{a^h - 1}{h}$ and $\frac{a^{-h} - 1}{-h}$.

91. When $a = 2$, the above rate is 0.693 . . . .

92. The varying rate is $2^x \times 0.693 = y \times 0.693$. The ratio $\frac{\text{rate of growth}}{\text{growing number}}$ is always 0.693 . . . .

93. Any number 2^b can be reached by unity's growth at a logarithmic rate r with regard to zero growing to $\frac{b}{r} \times 0.693$. The logarithmic rate r requires the base $2^{r \div 0.693}$. The logarithmic rate unity requires the natural base $2^{1 \div 0.693} = e = 2.71828$ . . . .

94. $\left(1 + \frac{1}{n}\right)^{n+1} > e > \left(1 + \frac{1}{n}\right)^n$; $\left(1 + \frac{x}{n}\right)^{n+x} > e^x > \left(1 + \frac{x}{n}\right)^n$. When n is large, $e^x = \left(1 + \frac{1}{n}\right)^{nx} = \left(1 + \frac{x}{n}\right)^n$.

95. Money put out at simple and compound interest, respectively, grows roughly at uniform and logarithmic rates.

VII. Graphs.

96. By a simple convention paired values of x and y determine a series of points (x, y), forming the graph of any given relation between x and y. The graphs of $y = \frac{1}{2}x$ and $y = 2^x$.

97. $\frac{y - y'}{x - x'}$ is the slope of the line joining (x, y) and (x', y'). $\frac{y' - y'}{x' - x'}$ is the slope of a graph at (x', y').

98. If, on the graph of $y = a^x$, the line through (x, y) and (x', y') cuts the line of x's where $x = k$, then, so long as $x - x' = h$, a constant, $x' - k = \frac{h}{a^h - 1}$, another constant. This gives a geometrical construction of the logarithmic curve and a geometrical interpretation of $e^x = \left(1 + \frac{x}{n}\right)^n$.

PART SECOND.

DOUBLE NUMBERS.

I. Integral Double Numbers and the Simpler Operations.

99. $\sqrt{-a^2} = ai$; $ai + bi = (a+b)i$; $ai \times bi \times ci = -abci$. The absolute value of a product of i and non-i numbers is the product of the absolute values of its factors; the product is an $\left\{ \begin{matrix} i \\ \text{non-}i \end{matrix} \right\}$ number if the number of i factors is $\left\{ \begin{matrix} \text{odd} \\ \text{even} \end{matrix} \right\}$; and, in determining its sign, each pair of i factors counts for a minus in addition to the minus signs before the factors

100. Non-i and i numbers are object and group numbers. The double number $a + bi$ marks the ath object in the bth group. Two double numbers are equal if their i and non-i parts are separately equal, and the sum of two double numbers is the number reached by counting from either, as we would count from zero to get the other.

101. The product of $a + bi$ by $c + di$ is the ath number in the bth group of the $c + di$ system. The groups of objects may be rows of dots. The laws for the multiplication of simple numbers hold for double numbers. Raising to integral non-i powers is done by repeated multiplication.

102. Subtraction is the addition of an opposite, and division is a guessing what to multiply one given double number by to get another.

II. Non-integral Double Numbers: Tensors and Sorts.

103. The interpolation of fractions and incommensurables gives Argand's diagram, $x + iy = (x, y)$.

104. Fractional double numbers are simple fractions of double integers. Double numbers may be of the same, of opposite, and of different sorts.

105. $(a+bi) \div (c+di) = (a+bi)(c-di) \div (c^2+d^2)$. $T(c+di) = T(c-di) = \sqrt{c^2+d^2}$.

106. Every double number is the product of a quantity and a quality factor.

107. If two numbers are of different sorts, the tensor of their sum is less than the sum but greater than the difference of their tensors.

108. The tensor of the $\left\{ \begin{matrix} \text{product} \\ \text{ratio} \end{matrix} \right\}$ of two numbers is the $\left\{ \begin{matrix} \text{product} \\ \text{ratio} \end{matrix} \right\}$ of their tensors. The tensor of a non-i power of a double number is the non-i power of the number's tensor.

109. One number lies between two others if, and only if, in some system or other the parts of the one lie between the parts of the others. $a + ib$ lies between $a_1 + ib_1$ and $a_2 + ib_2$ if, and only if, $(a_1 - a)(a - a_2) + (b_1 - b)(b - b_2) \not< 0$.

110. A number may be such with reference to two others that, in all systems, its parts lie between the parts of the two others.

III. Complex Units and Non-i Powers.

111. Because $\sqrt{p+iq} = \sqrt{\frac{1+p}{2}} + i\sqrt{\frac{1-p}{2}}$, it is possible to take of any double number a root whose index is an integral power of 2.
112. Any complex unit between unity and the doubly positive complex unit $p+qi$ can be expressed as closely as one pleases by a fractional power of $p+qi$.
113. All complex units are hemmed in as closely as one pleases by integral non-i powers of a doubly-positive complex unit whose i part is small.
114. The powers of the doubly-positive complex unit can be replaced by powers of a positive-negative complex unit.
115. Any complex unit is, as near as one pleases, a fractional power of any other complex unit. Conversely, a complex unit can be found that shall come as near as one pleases to any assigned non-i power of a double number. A non-i power of a non-unit double number is the product power-of-number's-tensor $\times$ power-of-complex-unit-of-number's-sort.

IV. Growths, Rates, and Amounts.

116. A double number grows by the separate growths of its i and non-i parts. Growths of double numbers are represented by graphs; uniform growths, by straight lines; varying growths, by curves.
117. If $u+iv=(c+id)(x+iy)$ and $c+id$ is constant, the growth of $u+iv$ is the same for the $c+id$ system that the growth of $x+iy$ is for the standard system.
118. If from each of two numbers there is a uniform growth, there will always be one and generally only one number reached by the growth. If more than one, then an infinity of numbers is reached.
119. All numbers directly between $a+ib$ and $a'+ib'$ are given by $l(a+ib)+l'(a'+ib')$ where $l>0$, $l'>0$, and $l+l'=1$.
120. A single uniform growth from $a+ib$ to $c+id$ is more direct than a chain of uniform growths from $a+ib$ to $c+id$ through numbers not directly between $a+ib$ and $c+id$.
121. If the chain of growths joining $a+ib$ to $c+id$ through x_1+iy_1, x_2+iy_2, . . . , x^n+iy^n, is such that $a<x_1<x_2<\ldots<x_n<d$ and $\frac{y_1-b}{x_1-a} > \frac{y_2-y_1}{x_2-x_1} > \ldots > \frac{d-y_n}{c-x_n}$, then a uniform growth joining two numbers on different growths of the chain cannot contain a third number on the chain.
122. A chain of growths of the same character as that of § 121, but through numbers all directly between $a+ib$ and numbers on that chain, is more direct than that is.
123. Numbers taken on a varying growth from $a+ib$ to $c+id$ such that $\frac{y-y}{x-x}$

decreases with increasing x, determines two chains of growths related like those of § 122.

124. By taking numbers on the varying growth closer and closer together, the $\left\{\begin{matrix}\text{more}\\ \text{less}\end{matrix}\right\}$ direct of the two chains determined by the two numbers becomes $\left\{\begin{matrix}\text{less and less}\\ \text{more and more}\end{matrix}\right\}$ direct, and the difference of the amounts of the two chains becomes as small as one pleases. Each amoun becomes the amount of the varying growth.

125. The total amount of any growth can be gotten by breaking it up into parts for which $\frac{y-y}{x-x}$ increases with increasing x, decreases with increasing x, or remains constant.

V. Logarithmic Growths and Double-number Powers.

126. When unity grows through all complex units around to unity again, the amount of growth is $2\pi = 2 \times 3.14159265 \ldots$

127. By non-i powering of i unity grows at the logarithmic rate $\frac{i\pi}{2}$; and by the powering of $i^{\frac{2}{\pi}}$, at the logarithmic rate i with regard to zero growing non-i-ward.

128. i is reached by unity's growth at the logarithmic rate unity with regard to zero growing i-ward to $\frac{i\pi}{2}$. A double-number power of a double number is a double number: $(e^k i^l)^{f+gi} = e^{kf-lg\frac{\pi}{2}} \cdot i^{lf+2\frac{kg}{\pi}}$.

129. $\left(1+\frac{i}{n}\right)^n = \left(1+\frac{i}{n}\right)^{ni} = e^i$, when $n = \infty$.

130. The numbers $1+\frac{p+qi}{n}$, $\left(1+\frac{p+qi}{n}\right)^2$, $\left(1+\frac{p+qi}{n}\right)^3$, ... are all on a growth from unity of the logarithmic sort $p+qi$, and the amount of growth from unity to $\left(1+\frac{p+qi}{n}\right)^k$ is $\frac{1}{p}\left(e^{\frac{kp}{n}}-1\right)$. Geometric illustrations.

131. Every number has k distinct kth roots. All these roots have the same tensor, but no two of them lie on the same logarithmic growth from unity. An incommensurable power of a number is, as near as one pleases, any number having the right tensor.

132. If $a+ib$ is a logarithm to base e of $c+id$, so also is $a+(2n\pi+1)b$, where n is any integer.

133. By proper choice of $p+iq$ the logarithmic growth $(p+iq)$-ward from one of two given numbers will contain the other.

134. The $c+id$ power of $a+ib$ is the result of unity's growing at the logarithmic rate $\log_e(a+ib)$ with regard to zero growing $(c+id)$-ward to $c+id$.

VI. Tensor Representation: Sines and Cosines.

135. When $e^a = r$, $e^{a+ib} = r_b$. A growth of r_b is determined by any relation between r and b.
136. If $1_b = p + iq$ and $1_{b+\frac{\pi}{2048}} = p' + iq'$, any intermediate complex unit, $1_{b+\frac{n\pi}{2048}}$, is very nearly $mp + np' + i(mq + nq')$, where $m + n = 1$.
137. $1_b = \cos b + i \sin b$. Unity $= 57° \; 17' \; 44''.8$.
138. Conclusion: Retrospect and prospect.

AN INTRODUCTION

TO THE

LOGIC OF ALGEBRA.

PART FIRST.

SIMPLE NUMBERS.

I. THE DIRECT OPERATIONS WITH POSITIVE INTEGERS.

1. The essential characteristic of mathematics is that all truths therein are established by reasoning. Suggested they may be by observation and confirmed by experiment; but finally settled they must be by rigid deduction, by showing that the statements to be proved are necessary consequences of other statements already known to be true or, at any rate, taken to be true.

When I say, "All snow is white; this is snow; therefore this is white," the mental process is so simple as almost to escape attention. Yet it is by the constant repetition of just such processes that the most profound researches in mathematics are carried on. The successive steps are easy; the difficulty lies in seeing what steps to take.

2. Reasoning must be carried on by language of some sort. Of course the more perfect the language, the more clearly it brings before the mind the statements needed and their relations; and the more completely it shuts out all irrele-

vant ideas, the greater the ease will be with which the mind takes in the reasoning.

The intricacy of mathematical reasoning has necessitated a language peculiar to that science. In the symbols and notation of arithmetic we have its beginnings. When from the consideration of particular numbers we pass to the consideration of numbers in general, the language grows. New terms are added and the meaning of the old ones is enlarged; *arithmetic, the art of computation, becomes algebra, the science of numerical relations.*

3. We are all familiar with the names one, two, three, . . . In counting a group of objects we apply to them, one by one, these names in order, till each object has its name. The last name given is the number of objects in the group. Manifestly this name is independent of the order of counting.

"Manifestly," did I say? How is it manifest? Did any one ever try it for all possible groups of objects? Can any one so try it? Do we not, after all, assume, because it has turned out true in the many cases in which we and others have tried it, that therefore it must always be true? The assumption is natural, perhaps justifiable; nevertheless it is altogether needless.

For, let k stand for any number whatever, and l for the next greater number. Imagine that a certain group of objects were counted in all possible orders, and that every count gave k for the number of objects in the group. Add an object to the group and count again. If this new object is counted after all the others, it takes the name or number next after k, that is l. If counted before some, then these take each a number next greater than one given them in some previous counting. In particular, the object then counted last and so called k is still counted last, but must now be called l.

Thus if, in counting any group, the last name or number given is independent of the order of counting, it remains so when we add an object to the group. But starting with a single object, and adding objects one by one, gives any group whatsoever. At the start k is one and l two; then k two and

l three, k three and l four, . . . : at first evidently, and so at each successive stage, and therefore finally, there is one and only one result of counting.

4. Suppose two groups of objects A and B. Let the number of objects in A be a; and in B, b. Count all the objects in both groups, beginning with the objects from A. The last object counted from A takes the number a, and so the first from B the number $a + 1$, a increased by one, the next $a + 2$, and so on; the last taking the number $a + b$, a increased by b. This is the total number of objects in both groups. But count first the objects from b and this same total number is ~~$a + b$;~~ $b + a$

$$\therefore \quad a + b = b + a.$$

The sum of two numbers is independent of the order of adding—*addition is commutative.*

5. Similarly, if there were a third group C of c objects, we should find

$$a + b + c = c + a + b = b + c + a = c + b + a = b + a + c = a + c + b.$$

Now $a + b + c$ means $(a + b) + c$, a-increased-by-b increased by c, and $b + c + a$ means $(b + c) + a$. But this last is the same as $a + (b + c)$;

$$\therefore \quad (a + b) + c = a + (b + c),$$
$$(2 + 3) + 7 \text{ or } 5 + 7 = 2 + (3 + 7) \text{ or } 2 + 10;$$

and the sum of three numbers does not depend at all upon which two of the three were first gathered into a partial sum.

In this proof we used commutation quite needlessly. For notice: in the expression $a + b + c$ we think of the objects in A as named from 1 to a, of those in B as named from 1 to b, and of those in C as named from 1 to c. In $(a + b) + c$ the objects in B are named from $a + 1$ to $a + b$; and in $a + (b + c)$ the objects in C are named from $b + 1$ to $b + c$. But pass from either $(a + b) + c$ or $a + (b + c)$ to the final sum, and the objects in A are named from 1 to a, those in B from $a + 1$ to $a + b$, and those in C from $a + b + 1$ to $a + b + c$. The final

result hangs only upon the names finally given and not at all upon the various changes of name taken by the objects in getting that result.

The same reasoning shows that the manner of grouping does not affect the sum of four, five, any number of numbers; i.e., *addition is associative.*

The student may prove that if the associative law holds for the sum of k numbers, it must hold for $k + 1$ numbers and therefore universally. Also, he may show that the sum of any number of numbers is independent of the order of adding.

6. Suppose b groups of a objects each. The total number of objects in the groups is

$$a + a + a + a + a + \ldots \text{ to } b \; a\text{'s}.$$

We call this sum the product of a by b and write it $a \times b$, a multiplied by b.

From each of the b groups take an object; they together form a group of b objects: another object from each of the groups, and we have another group of b objects. In all there are a such groups of b objects each. Together they contain $b + b + b + b + \ldots$ to a b's or $b \times a$ objects.

$$\therefore \quad a \times b = b \times a.$$

$$\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array}$$

$$3 \times 4 = 4 \times 3.$$

In words, the product of two numbers is independent of the order of the factors—*multiplication is commutative.*

7. Just as we understand $a + b + c$ to mean $(a + b) + c$, so we understand $a \times b \times c$ to mean $(a \times b) \times c$. This is the number of objects in c groups of b-groups-of-a-objects. In each of the c groups there are b sub-groups, and in all the c groups $b \times c$ sub-groups. As each of these $b \times c$ sub-groups contains a objects, the number in all of them is $a \times (b \times c)$.

$$\therefore \quad (a \times b) \times c = a \times (b \times c);$$

and for the product of any three numbers, *multiplication is associative.*

E.g., :: :: ::

:: :: ::

$$(4 \times 2) \times 3 \text{ or } 8 \times 3 = 4 \times (2 \times 3) \text{ or } 4 \times 6.$$

8. Because multiplication is both commutative and associative,

$$a \times b \times c = b \times c \times a = c \times a \times b = c \times b \times a = b \times a \times c = a \times c \times b,$$

and the product of three numbers is independent of the order of the factors.

Suppose that the product of any number of factors up to n inclusive is independent of the order of the factors; then also is the product of $n + 1$ factors.

If there is any change in the final result, it must be due to starting with different pairs of factors; for the moment we multiply together any two of the $n + 1$ factors, we have then to deal with but n factors, the pair-product and $n - 1$ other factors.

Let a, b, c be any three out of the $n + 1$ factors. Starting with $a \times b$, we can go on as we please and so have $a \times b \times c$. Likewise, starting with $a \times c$, we can take for the first three $a \times c \times b$. But $a \times b \times c = a \times c \times b$, and so the product of the $n + 1$ factors is the same starting with $a \times b$ as starting with $a \times c$. If, however, we can change one of the factors in the first pair, we can the other also; i.e., the first pair of factors can be any two out of the $n + 1$ factors. Thus the proposition is proved.

But the product of two factors, and also that of three factors, is independent of the order of the factors: then, too, is that of four factors, of five, six, any number of factors.

9. In getting the final product the factors can be grouped into partial products in any way we please.

E.g., $a \times b \times c \times d \times e \times f = a \times (b \times c \times d) \times (e \times f).$

For the expression on the right is

$$\begin{aligned}&(b \times c \times d) \times a \times (e \times f) = b \times c \times d \times a \times (e \times f)\\ &= (e \times f) \times b \times c \times d \times a = e \times f \times b \times d \times e \times f\\ &= a \times b \times c \times d \times e \times f;\end{aligned}$$

or, without using the commutative law,

$$\begin{aligned}&a \times (b \times c \times d) \times (e \times f) = a \times [(b \times c) \times d] \times (e \times f)\\ &= a \times [b \times (c \times d)] \times (e \times f) = (a \times b) \times (c \times d) \times (e \times f)\\ &= (a \times b \times c) \times d \times (e \times f) = (a \times b \times c \times d) \times (e \times f)\\ &= (a \times b \times c \times d \times e) \times f = a \times b \times c \times d \times e \times f.\end{aligned}$$

The general proof of this, the *associative law for multiplication*, will be an excellent exercise for the student.

10. We wrote for $a + a + a + \ldots$ to b a's, $a \times b$. We now write for $a \times a \times a \times \ldots$ to b a's, a^b, and call the expression the bth power of a. a is the *base*, b the *index* of the power, and b is the *exponent* of a. In getting the expression, a is said to be raised to the bth power or to be powered by b, and the process is called *involution*.

11. Base and index cannot in general be interchanged. E.g., $2^3 = 8$, but $3^2 = 9$; $2^5 = 32$, but $5^2 = 25$. Nevertheless, $2^4 = 4^2 = 16$. By trying a number of cases the student can probably satisfy himself that $2^k \neq k^2$ unless $k = 2$ or 4; then, with slightly more difficulty, that $3^k \neq k^3$ unless $k = 3$. Later he may be able to see under just what conditions $a^b = b^a$. A single case of failure, however, suffices to show that *for involution there is no commutative law.*

On the other hand, a thousand successes, even though we had not come upon a failure, would not have proved the law. They would only create a presumption in its favor and make it worth our while to look further: to inquire into the reason of the successes, and see if that reason must hold in all cases and so necessitate the law.

12. *Involution is non-associative.* For $2^{(3^2)} = 2^9 = 512$, but $(2^3)^2 = 8^2 = 64$. Here, again, there is not always failure. Thus

$(3^2)^2 = 9^2 = 81$, and $3^{(2^2)} = 3^4 = 81$; $(5^2)^2 = 25^2 = 625$, and $5^{(2^2)} = 5^4 = 625$.

13. One property, however, involution shares with addition and multiplication. All three are uniform processes. That is to say, the results of the processes cannot be changed without changing the numbers used in getting the results. In symbols— $a = a'$ and $b = b'$ require $a + b = a' + b'$, $a \times b = a' \times b'$, and $a^b = a'^{b'}$.

14. We now consider expressions in which more than one of the above fundamental processes occur.

To avoid the useless writing of parentheses we agree that the parts of an expression separated by + signs, the terms of the expression, are to be first calculated and then the results added. Addition is said to take *precedence* of both multiplication and involution. Thus $7 + 2 \times 3 + 5^2 = 7 + 6 + 25 = 38$.

In the same way, multiplication takes precedence of involution: $3 \times 5^2 = 3 \times 25 = 75$. The multiplications are performed upon the results of the involutions, the additions upon the results of the multiplications; hence the use of the word 'precedence.'

Whatever is connected with the exponent of a power by any sign forms part of that exponent. a^{b+c} means $a^{(b+c)}$; $a^{b\times c}$, $a^{(b\times c)}$; and a^{b^c}, $a^{(b^c)}$.

Notice that all the above is arbitrary; other agreements as to our use of mathematical language might be made, sometimes have been made. We merely follow prevailing usage, a usage that has come about, as most changes in language come about, from the attempt to express ideas with as little trouble as possible.

Let the student calculate the value of these expressions:

$$2^{2\times 2^2}, \qquad 2^{1+1^{1+1}}, \qquad 3^{2^{1^2}}, \qquad 4^{2+1^{2\times 3}\times 2} + 5 \times 3^{2^3}.$$

15. Consider the expression $b \times a + c \times a$. It denotes the number of objects in a groups of b objects each, together with the number of objects in a groups of c objects each. With

each of the groups of b objects we can place a group of c objects, thus forming a groups of $(b+c)$ objects. In these there are $(b+c) \times a$ objects.

$$\therefore \quad b \times a + c \times a = (b+c) \times a.$$

Moreover, since multiplication is commutative,

$$a \times b + a \times c = a \times (b+c).$$

In the expressions on the left the product is said to be *distributed;* and we say *multiplication is distributive with regard to addition whether the sum be multiplier or multiplicand.*

The proof is easily pictured to the eye, thus:

$$\begin{array}{cccccc} \bullet & \bullet & & \bullet & \bullet & \bullet \\ \bullet & \bullet & & \bullet & \bullet & \bullet \\ \bullet & \bullet & & \bullet & \bullet & \bullet \end{array}$$

$$2 \times 3 + 3 \times 3 = (2+3) \times 3 = 5 \times 3.$$

The student may prove that

$$(a+b) \times (c+d) = a \times c + a \times d + b \times c + b \times d;$$

and, in general, the product of two sums of numbers is the sum of all the products obtained by multiplying each number of the one sum by each number of the other.

Then let him state and prove a rule for the product of three or more sums.

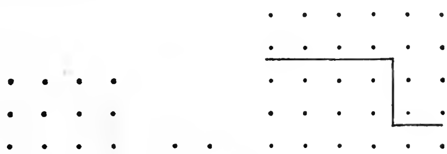
When from the distributed product we pass back to the single term, from $a \times b + a \times c$ to $a \times (b+c)$, the terms $a \times b$ and $a \times c$ are said to be *collected*, *summed*, or *added.*

16. We have seen that both addition and multiplication are commutative and associative. We have just now established a relation between the two processes. Is this relation reciprocal? is addition distributive with regard to multiplication? In

$$(b+c) \times a = (b \times a) + (c \times a)$$

can the signs '$+$' and '$\times$' be interchanged and the equality hold? (Notice the introduction of parentheses above on the right and explain it.)

We have $(b+a) \times (c+a) = b \times c + a \times c + b \times a + a^2$; but plainly $(b \times c) + a$ cannot exceed the first two terms of this; and therefore *addition is non-distributive with regard to multiplication*, the proof not being changed in character when the product is the second instead of the first term of the sum.



$$3 \times 4 + 2 \neq (3+2) \times (4+2).$$

17. In order to more easily see the relations between involution and the other processes, adopt for the time being a notation for involution similar to that for multiplication, writing

$$a^b = a \,\mathrm{p}\, b,$$

where the p may be read 'powered by.'

At once, then, in order that involution shall be completely distributive with regard to multiplication requires that both in

$$(b+c) \times a = (b \times a) + (c \times a)$$

and in
$$a \times (b+c) = (a \times b) + (a \times c)$$

we shall be able, keeping the equality true, to write p and $\times$ for $\times$ and $+$.

The first gives

$$(b \times c) \,\mathrm{p}\, a = (b \,\mathrm{p}\, a) \times (c \,\mathrm{p}\, a),$$

or
$$(b \times c)^a = b^a \times c^a,$$

a true equality; for

$$\begin{aligned}(b \times c)^a &= (b \times c) \times (b \times c) \times (b \times c) \times \ldots \text{ to } a \ (b \times c)\text{'s} \\ &= (b \times b \times b \times \ldots \text{ to } a \ b\text{'s}) \times (c \times c \times c \times \ldots \text{ to } a \ c\text{'s}) \\ &= b^a \times c^a.\end{aligned}$$

Hence, *involution is distributive with regard to multiplication when the product is the base.*

When the product is the index the distributive law would require

$$a \mathrm{p} (b \times c) = (a \mathrm{p} b) \times (a \mathrm{p} c),$$

or $$a^{b \times c} = a^b \times a^c.$$

This, however, is a false equality, for

$$\begin{aligned} a^b \times a^c &= (a \times a \times a \times \ldots \text{to } b\ a\text{'s}) \times (a \times a \times a \times \ldots \text{to } c\ a\text{'s}) \\ &= a \times a \times a \times \ldots \text{to } (b + c)\ a\text{'s} \\ &= a^{b+c} \neq a^{b \times c}, \text{ unless } b = c = 2. \end{aligned}$$

Consequently, *involution is non-distributive with regard to multiplication when the product is the index.*

18. In order that multiplication shall be distributive with regard to involution we need

$$(b \mathrm{p} c) \times a = (b \times a) \mathrm{p} (c \times a)$$

and $$a \times (b \mathrm{p} c) = (a \times b) \mathrm{p} (a \times c);$$

that is to say, we must have

$$b^c \times a = (b \times a)^{c \times a}$$

and $$a \times b^c = (a \times b)^{a \times c};$$

both which equalities are false unless $a = 1$. Therefore, *multiplication is non-distributive with regard to involution.*

19. *Involution is non-distributive with regard to addition when the sum is the base.*

$$(a + c) \mathrm{p} a \neq (b \mathrm{p} a) + (c \mathrm{p} a)$$

or $$(b + c)^a \neq b^a + c^a.$$

For, unless $a = 1$, we shall get, on expanding $(b + c)^a$, other terms in addition to b^a and c^a.

Neither is it distributive when the sum is the index.

$$a \mathrm{p} (b + c) \neq a \mathrm{p} b + a \mathrm{p} c,$$

$$a^{b+c} \neq a^b + a^c.$$

For we have already seen that $a^{b+c} = a^b \times a^c$, and $a^b \times a^c \neq a^b + a^c$ unless $a = 2$ and $b = c = 1$.

20. *Addition is non-distributive with regard to involution.*

$$b^c + a \neq (b+a)^{c+a}, \quad \text{and} \quad a + b^c \neq (a+b)^{a+c}.$$

Any exception?

21. Since $$a^b \times a^c = a^{b+c},$$

it follows that $a^{b \times c}$ or $a^{b+b+b+b+\dots \text{ to } c\, b\text{'s}}$ is

$$a^b \times a^b \times a^b \times a^b \times \dots \text{ to } c\ a^b\text{'s} = (a^b)^c;$$

and thence, $$a^{b \times c \times d \times e} = \{[(a^b)^c]^d\}^e.$$

It is because $(a^b)^c$ is expressed by $a^{b \times c}$ that we use a^{bc} to mean $a^{(bc)}$.

By § 17, *the product of the same powers of several bases is the product-of-the-bases raised to the common power.*

This is the *distributive law for involution.*

Also, *the product of powers of a common base is that base powered by the sum of the given indices.*

We now see that *a power is itself raised to a power by multiplying its index by the index of the power to which it is to be raised.*

These two are *the laws of the power index.*

22. Though involution is non-commutative, yet it has a property resembling commutation. Using the p-notation and agreeing that the operations denoted by a succession of p's shall be performed in order from left to right, we have

$$\{[(a^b)^c]^d\}^e = a\,\mathrm{p}\,b\,\mathrm{p}\,c\,\mathrm{p}\,d\,\mathrm{p}\,e.$$

Now, if in these expressions we keep a first, the other letters may be put in any order we please; for the changed expressions would, like the original ones, all be

$$a\,\mathrm{p}(b \times c \times d \times e) = a^{b \times c \times d \times e}.$$

II. The Inverse Operations with Positive Integers.

23. Not only is it true that the three fundamental processes are *uniform*, giving with the same numbers the same results; but also it is true that if either of the numbers combined in the operations is changed the results will be changed. Thus,

$$b > c \text{ requires } a + b > a + c,\ a \times b > a \times c,\ a^b > a^c,\ b^a > c^a.$$

There is one exception: $1^b = 1^c$ even though $b > c$.

The problem, given one of the numbers determining a result of one of the three processes, to find the other number, leads to the four new processes of subtraction, division, evolution, and taking logarithms.

24. *Subtraction is the process that undoes addition.* Thus, if $a + b = c$, then $c - b = a$, c diminished by b is a. But, by commutation, $b + a = c$, and so $c - a = b$.

The number after the minus sign is said to be *subtracted from* the number before it. The sum is now called the *minuend*, the number subtracted the *subtrahend*, and the result the *remainder*.

25. *Division is the process that undoes multiplication.* Accordingly, if $a \times b = c$, then $c \div b = a$; and because of commutation, $c \div a = d$. The number before the sign $\div$ is said to be divided by the number after it; or, if we prefer, we may say that the number after the sign is *divided out of* the number before it. So $\times$ is sometimes read *multiplied into.* The product that we divide is called the *dividend*, the number dividing the *divisor*, and the result the *quotient*.

26. Since involution is non-commutative, it may be undone in two ways: so as to determine an unknown base or so as to determine an unknown index.

27. *The undoing of involution that gives the base is called evolution.* When $a^b = c$, we write $\sqrt[b]{c} = a$, and read, "the bth root of c is a." c is called the *root-base* and b the *root-index.* The root-index 2 is commonly omitted.

28. *The undoing of involution that gives the index is called taking the logarithm.* Thus, $a^b = c$ gives $b = \log_a c$, *b is the logarithm to the base a of c.* In the early development of algebra this process was overlooked, and so has come to be classed as *non-algebraic* or *transcendental,* the other processes being called *algebraic.*

29. If, starting with any number, we perform a series of operations upon it and thus get another number, we can of course get back to the starting number by merely undoing these operations in an order the reverse of that in which they were performed.

E.g., if $$[(a+b)\times c]^d = e,$$

then $$a = [(\sqrt[d]{e}) \div c] - b, \quad b = [(\sqrt[d]{e}) \div c] - a,$$
$$c = (\sqrt[d]{e}) \div (a+b), \quad d = \log_{[(a+b)\times c]} e;$$

all which equations may be verified by substituting numbers for the letters.

In like manner, given the equalities below, let the student express each letter in terms of the others.

$$\sqrt[g]{[(\sqrt[a]{b} \div c) \div d]^e - f} = h;$$

$$\left[\{[\sqrt[a]{(c-e} \times f)^b] \div g\} + h\right]^k = l\cdot$$

$$(a\times b + c)^d = [(e \div f) - g]^h.$$

It will be well for him to test his results by the substitution of numbers for letters. He can easily devise for himself as many more problems of this kind as he pleases.

30. The first three processes that we took up are called direct; the last four are called inverse. In particular, evolution is the *first inverse,* and taking a logarithm the *second inverse* of involution.

The actual performance of the inverse operations is a guessing and trying founded upon previous knowledge of the results of the direct operations.

We know that $12 - 5 = 7$ because $7 + 5 = 12$; that $56 \div 7 = 8$ because $8 \times 7 = 56$; that $\sqrt{144} = 12$ because $12^2 = 144$; that $\log_3 81 = 4$ because $3^4 = 81$.

Consider the division of 2461 by 23.

$$\begin{array}{l} 23)\,2461\,(\,107 \\ \quad\;\, \underline{23} \\ \qquad 161 \\ \qquad \underline{161} \end{array} \qquad \begin{array}{r} \text{Because } 200 \times 23 = 4600 > 2461, \\ \text{and } 100 \times 23 = 2300 < 2461, \\ \therefore \quad 200 > 2461 \div 23 > 100. \end{array}$$

Again, because $2461 - 2300 = 161$, and 161 is less than 10×23, $\therefore 2461 < 110 \times 23$. Because 20, which is smaller than 23, would be contained barely 8 times in 161, we guess 7 for the number of times that 23 is contained in 161. Our guess proves right, and consequently $2461 \div 23 = 107$.

After this fashion can be analyzed any example in division.

In mathematics, as in other sciences, guessing and trying are the two most important tools that the student has. Let him not shrink from their use.

In the following equations he may guess values of x and y that will make the equations true.

$x^2 + 7 \times x = 18$; $2 \times x^3 - 3 \times x^2 = 4$; $3^x = x^3$; $\log_2 x = 64$; $\log_x 16 = 2$; $5 \times x + 3 \times y = 8$; $5 \times x + 3 \times y = 19$; $3 \times x + 7 \times y = 58$; $11 \times x - (\sqrt{y}) = 12$; $x^2 + y = 7$; $x^y = 2$; $\sqrt[y]{x} = 3$; $y^x = 64$; $\sqrt{x + y} = 3$; $\sqrt[3]{x^2 + y^2} = x - y$.

Sometimes in the above he may find more than one value or set of values.

31. We return to the inverse operations. Are they commutative? Try it. We have $8 - 2 = 6$; $12 \div 4 = 3$; $\sqrt[3]{8} = 2$; $\log_2 16 = 4$; but $2 - 8$, $4 \div 12$, $\sqrt[8]{3}$, $\log_{16} 2$, have as yet no meaning.

Hence, *the inverse operations are not commutative.*

32. *Nor are they associative.* Thus: $(8 - 4) - 2 = 4 - 2 = 2$, but $8 - (4 - 2) = 8 - 2 = 6$; $(24 \div 4) \div 2 = 6 \div 2 = 3$, but $24 \div (4 \div 2) = 24 \div 2 = 12$; $\sqrt{(\sqrt[4]{256})} = \sqrt{4} = 2$, but $\sqrt[(\sqrt{4})]{256} = \sqrt{256} = 16$; $\log_2 (\log_4 4096) = \log_2 8 = 3$, but $\log_{(\log_2 4)} 4096 = \log_2 4096 = 64$.

The student may have some difficulty in seeing that $\sqrt[a]{(\sqrt[b]{c})}$ and $\sqrt[(\sqrt[a]{b})]{c}$, $\log_a(\log_b c)$ and $\log_{(\log_a b)} c$, differ as to form in the same way that $(a+b)+c$ and $a+(b+c)$ do. The notation is somewhat confusing. To make matters plainer, just as we wrote $a^b = a \text{ p } b$, we will, for the moment, write $\sqrt[a]{b} = b \text{ up } a$, b 'unpowered' by a; then $\sqrt[a]{(\sqrt[b]{c})}$ is $(c \text{ up } b) \text{ up } a$, while $\sqrt[(\sqrt[a]{b})]{c}$ is $c \text{ up} (b \text{ up } a)$, and the analogy is manifest.

Again, if in a^b we regard b as changed into a new number by writing a to the left and below it, we can speak of b as 'based' by a, and write $a^b = b \text{ b } a$; with which goes $\log_a b = b \text{ ub } a$, b 'unbased' by a. So we get $\log_a(\log_b c) = (c \text{ ub } b) \text{ ub } a$, and $\log_{(\log_a b)} c = c \text{ ub } (b \text{ ub } a)$, all difficulty vanishing.

If the student is unsatisfied with the disproof just given of the commutative and associative laws for the inverse processes, he will find it not very difficult, after going on a little farther, to work out a proof based upon the definitions of the processes. Thus will he show not merely that the laws do fail, but also why they fail.

33. In the various expressions considered in the immediately foregoing paragraphs, certain parentheses are rendered unnecessary by the following conventions:

In expressions containing the signs $+$, $-$, $\times$, $\div$, $\sqrt{\ }$, log, $(\ )^{(\ )}$, the precedence is—

1st, $+\ -$; 2d, $\times\ \div$; 3d, $\sqrt{\ }$ log; 4th, $(\ )^{(\ )}$.

A chain of operations denoted by $+$ and $-$ signs are performed in order as if all the signs were $+$; a chain denoted by $\times$ and $\div$ signs as if all the signs were $\times$; a chain denoted by $\sqrt{\ }$ and log signs as if all the signs were $\sqrt{\ }$.

Since the symbols $\sqrt{\ }$ and log are written to the left of the numbers on which they operate, the operations denoted by a chain of them are performed in order from right to left.

Whatever is connected with a root-index or logarithm-base by any symbol whatever forms part of that root-index or logarithm-base.

Thus, $8-4-2$ is the same as $(8-4)-2$; $24 \div 4 \div 2$, as $(24 \div 4) \div 2$; $\sqrt{\sqrt[4]{256}}$, as $\sqrt{(\sqrt[4]{256})}$; $\sqrt[\sqrt{4}]{256}$, as $\sqrt[(\sqrt{4})]{256}$; $\log_2 \log_4 4096$, as $\log_2 (\log_4 4096)$; $\log_{\log_2 4} 4096$, as $\log_{(\log_2 4)} 4096$.

Again, $\log_2 \sqrt{64} = \log_2 8 = 3$; $\sqrt{\log_2 16} = \sqrt{4} = 2$; $\log_2 4^2 = \log_2 16 = 4$; $(\log_2 8)^2 = 3^2 = 9$; $\sqrt[3]{2^6} = \sqrt[3]{64} = 4$; $(\sqrt[3]{8})^2 = 2^2 = 4$.

The student may remove unnecessary parentheses from the expressions in § 29.

III. Negatives and Fractions.

34. We have seen that in certain cases the inverse operations failed. Algebra takes up these failures, compares them, reasons about them, operates with them,—in short, converts them into numbers. Thus, she "marshals victory out of defeat," and presses on to new and ever-widening fields of conquest.

Consider, now, her methods.

$5-8$ is to be a new sort of number as are likewise $6-9$ and $6-10$. All three represent attempts to take away more units than there are to take. In $5-8$ and $6-9$ we are asked to take away 3 too many; in $6-10$, 4 too many. We say then

$$5-8=6-9, \quad \text{but} \quad 5-8>6-10.$$

To obtain a general test of equality and inequality, suppose a, b, c, d to be four numbers; and first, let $a>b$ and $c>d$. Then $a-b$ is some ordinary number, say k, and $c-d$ is also an ordinary number, say g;

$$\text{i.e.,} \qquad a-b=k, \qquad \text{and} \qquad c-d=g.$$

$$\therefore \qquad a=k+b, \qquad \text{and} \qquad c=g+d;$$

whence $a+d=k+b+d$, and $b+c=g+b+d$.

Plainly, $a+d \gtreqless b+c$ if, and only if, $k \gtreqless g$; i.e., if

$$a-b \gtreqless c-d.$$

Thus, *when* $a - b$ *and* $c - d$ *are ordinary numbers,* $a + d \gtreqless b + c$ *is the necessary and sufficient condition that* $a - b \gtreqless c - d.$

We impose the same conditions on the new sort of numbers; and thus, *always* $a + d \gtreqless b + c$ *requires* $a - b \gtreqless c - d.$

Let the student prove that $a - b \gtreqless c - d$ if $b - a \lesseqgtr d - c$.

35. In particular, suppose $a = b$ and $c = d$; then $a + c = b + d$, and so $a - b = c - d$. But $a - b$ is $a - a$, and $c - d$ is $c - c$.

Hence, $\qquad a - a = c - c = 0$, say,

no matter what two numbers a and c may be. This is a new number, *zero or naught, the symbol of nothing to count.*

Ordinary numbers, 1, 2, 3, 4, . . . , may be conceived as gotten by adding to zero, and we may write them $0 + 1$, $0 + 2$, $0 + 3$, $0 + 4$, . . . ; or, for short, $+1$, $+2$, $+3$, $+4$, . . . We call them *positive* numbers.

If $$a - b = 0 + c,$$

then $$b - a = 0 - c;$$

for the test gives, writing b, a, 0, and c, in place of a, b, c, and d, respectively,

$$b + c = 0 + a = a,$$

a direct consequence of $a - b = c$.

Thus, to any positive number c there corresponds a number gotten by subtracting c from zero. Furthermore, the correspondence is reciprocal, and all the new numbers can be so gotten. We write them $0 - 1$, $0 - 2$, $0 - 3$, $0 - 4$, . . . , or, for short, -1, -2, -3, -4, . . . We call them *negative* numbers.

The number zero, neither positive nor negative, is the link between the two sorts of numbers.

The signs $+$ and $-$, originally used to denote addition and subtraction, are now used to show whether a number is the result of an addition or a subtraction from zero.

This double signification of the signs need not confuse us: the distinction is merely one of point of view. Precisely similar is that between $a+b$ regarded as a command to add b to a, or as the single number resulting from the addition.

The value of a number without regard to its sign, or say with its sign taken positive, is called its *absolute* value; and numbers whose absolute value is the same, but which differ in sign, are called *opposite* numbers.

36. We now have the series of names,

$$\ldots -4, -3, -2, -1, 0, +1, +2, +3, +4, \ldots$$

extending ad libitum *on forward* and *off backward* from zero.

A positive number is simply a name to which we come when we count on forward from zero; a negative number, a name to which we come when we count off backward from zero.

One number in the series is larger than another, if we have to count off backward to get from it to the other; and smaller, if we have to count on forward.

Opposite numbers are equally removed opposite ways from zero, and their absolute value is the number of their removes from zero.

37. In somewhat the same way that we compared the expressions $5-8$, $6-9$, and $6-10$, we can compare expressions like $a \div b$.

When b exactly divides a, we know that the larger a is, b remaining unchanged, the larger is $a \div b$; and, on the other hand, that when a is unchanged, the larger b, the smaller is $a \div b$.

Thus, $\quad 15 \div 3 > 12 \div 3, \quad$ and $\quad 15 \div 5 < 15 \div 3.$

Further, multiplying a multiplies $a \div b$; while multiplying b, leaving the division exact, divides $a \div b$.

For, suppose $a \div b = c$, $c \div d = e$, and f is any number; then $a \times f \div b = f \times a \div b = f \times (c \times b) \div b = f \times c \times b \div b$

$=f\times c=f\times(a\div b)$, which proves the first part of the statement.

Again, $a=b\times c$, but $c=d\times e$;

and so, $\quad a=b\times(d\times e)=e\times(b\times d)$.

$$\therefore\quad a\div(b\times d)=e\times(b\times d)\div(b\times d)=e$$
$$=c\div d=(a\div b)\div d,$$

which proves the other part.

It follows at once that if k be any number,

$$(a\times k)\div(b\times k)=a\div b;$$

for, $\quad (a\times k)\div(b\times k)=a\times k\div(k\times b)$
$$=a\times k\div k\div b=a\div b.$$

38. Suppose two expressions $a\div b$ and $c\div d$. Moreover, let the division be exact.

By what we have just proved,

$$a\div b=(a\times d)\div(b\times d),$$

and $\quad c\div d=(b\times c)\div(b\times d)$.

Necessarily, then,

$$a\div b \gtreqless c\div d, \text{ if } a\times d \gtreqless b\times c.$$

This, the *test* of equality when division is exact, we make the *definition* of equality when division is not exact. Thus the test becomes universal.

39. The new sort of numbers whose equality we have just defined are *fractions*, and we write

$$a\div b=\frac{a}{b},$$

where a is the *numerator* and b the *denominator* of the fraction.

In contradistinction the numbers with which we have previously been dealing are *whole* numbers or *integers*.

40. Whatever is connected with the numerator or denominator of a fraction by any symbol whatever forms part of that numerator or denominator, the bar of the fraction serving as a sign of inclusion.

$$\text{E.g., } \frac{\sqrt{4}}{3 \times 4 - 1} = \frac{2}{11};\ \frac{\log_2 8}{3^2} = \frac{3}{9};\ 4 \div \frac{12}{4} = 4 \div 3;$$

$$\sqrt{\frac{32}{2}} = \sqrt{16} = 4;\ \log_3 \frac{162}{2} = \log_3 81 = 4;\ \frac{5^2}{7} = \frac{25}{7}.$$

Thus powering is the only operation upon a fraction for which a sign of inclusion is needed.

41. Along with the new division notation it is convenient to bring in an abbreviated multiplication notation.

Instead of $\times$ we write . , or even merely write our letters and expressions together.

$$\text{Thus, } 2 \times 3 \times 5 \times a \times b \times (c + d) = 2 \,.\, 3 \,.\, 5\, ab\, (c + d).$$

Notice, in passing, that $3\frac{2}{8}$ means $3 + \frac{2}{8}$, but $a\frac{b}{c}$ means $a \times \frac{b}{c}$. In arithmetic $+$, in algebra $\times$, is omitted.

When multiplication is denoted by mere writing together of the factors, all operations, save only involution and evolution, take precedence of it.

Accordingly,

$$24 \div 2 \,.\, 3 = 24 \div 2 \times 3 = 36;\ \text{but } abc \div bc = a;$$

$$\log_c ab = \log_c (ab);\ \log_c a(b + c) = \log_c [a(b + c)].$$

There is division of usage as to the meaning of $\sqrt{ab}$, some making it $\sqrt{\overline{ab}}$, others $\sqrt{a} \,.\, b = b\sqrt{a}$. The latter usage is more general.

If the first be adopted, $\sqrt[3]{a}\sqrt[3]{b}$ ought to mean $\sqrt[3]{a\sqrt[3]{b}}$; if the other, $\sqrt[3]{a}\sqrt[3]{b}$ would mean $\sqrt[3]{a}\times\sqrt[3]{b}$.

So $\log_c a \log_c b$ ought to mean $\log_c(a \log_c b)$; and for the product of two logarithms, we should write $\log_c a \,.\, \log_c b$.

42. If $\frac{a}{b}$ and $\frac{c}{d}$ are two fractions such that $\frac{a}{b} > \frac{c}{d}$, then the fraction $\frac{a+c}{b+d}$, whose numerator is the sum of their numerators and whose denominator is the sum of their denominators, is intermediate in value to the two fractions;

i.e.,
$$\frac{a}{b} > \frac{a+c}{b+d} > \frac{c}{d}.$$

For, $\frac{a}{b} > \frac{c}{d}$ gives $ad > bc$, whence $ab + ad > ab + bc$; or, what is the same thing,

$$a(b+d) > b(a+c), \quad \text{and so} \quad \frac{a}{b} > \frac{a+c}{b+d}.$$

But $ad > bc$ also requires

$$(a+c)d > (b+d)c, \quad \text{and hence} \quad \frac{a+c}{b+d} > \frac{c}{d}.$$

In like manner, $\frac{2a+c}{2b+d}$ lies between $\frac{a}{b}$ and $\frac{a+c}{b+d}$; while $\frac{a+2c}{b+2d}$ lies between $\frac{a+c}{b+d}$ and $\frac{c}{d}$.

Plainly, we can go on forever finding intermediate fractions. In other words, between any two unequal fractions lie an infinite number of other fractions.

Find by the above method all the fractions of which neither the numerators nor the denominators exceed 10, and which lie between $\frac{1}{10}$ and 10. Arrange them in the order of their magnitude.

Given a set of fractions, prove that any new fraction

$\dfrac{\text{sum of multiples of numerators}}{\text{sum of multiples of denominators}}$ is smaller than the greatest and larger than the least fraction of the set.

If all the fractions of the set are equal, how about the value of the new fraction?

The positive integers 1, 2, 3, 4, . . . are fractions whose denominators happen to exactly divide their numerators. In particular, then, between any two positive integers lie an infinite number of fractions.

Analogy suggests that we ought also to have fractions lying between negative integers. We shall presently see how these arise.

43. We now extend the meaning of the words 'addition' and 'sum' so as to apply them to negative integers.

The sum of two integers is the integer gotten by counting $\left\{\begin{matrix}\textit{on forward}\\ \textit{off backward}\end{matrix}\right\}$ *from either as we would count* $\left\{\begin{matrix}\textit{on forward}\\ \textit{off backward}\end{matrix}\right\}$ *from zero to get the other.*

To get the sum of several integers, we take the sum of any two of them, the sum of that sum and a third, of that and a fourth, and so on—until all of the integers have been used.

Addition is the process of getting this sum. Subtraction, as before, is, now and always, the process that undoes addition.

44. It follows directly from the definition that we can indicate the addition of integers by merely writing them in any order connected by their proper signs.

E.g., the sum of -7, $+4$, -2, and $+1$ is

$$-7+4-2+1, \quad \text{or} \quad 4-7+1-2, \quad \text{or} \quad 1-2+4-7.$$

We do not say that these sums are the same; we do not say that these are the only sums; we merely say that either of the above three expressions could, in perfect accordance with our definition, be the sum of the four given numbers.

Since to count from an integer to zero requires the same number of counts and in the same direction as to count from zero to the opposite of the number, it follows that to subtract an integer is the same as to add its opposite.

45. When all the numbers added happen to be positive, addition as just defined falls in with ordinary addition. When necessary to distinguish it from ordinary addition we call it *algebraic*, and the sum is an *algebraic sum*.

With algebraic as with ordinary or arithmetical addition, the order of adding is indifferent.

For, consider the sum

$$a - b - c + d - e + f.$$

This is

$$1+1+1+\dots\underline{+1, -1}-1-1-1-\dots-1, -1-1-1-\dots-1, \dots$$

or, a counts forward, b backward, c backward, . . . ,

and we count

$$1, 2, 3, \dots a-1, \underline{a; a-1}, a-2, \dots a-b, a-b-1, \dots a-b-c; \dots$$

The final count is in nowise changed if we remove the counts a, $a - 1$ underlined above. This is the same as if, in the unit additions, we removed the underlined combination $+1-1$, at the first change of sign. Of course the combinations $+1-1$ and $-1+1$ can be removed wherever they occur. To keep doing so will finally get the signs all of one kind. The number of units left will be the excess of the number of units of one kind over the number of units of the opposite kind. But this excess depends merely upon the number of units of each kind there were at the start and not at all upon their order.

Hence, *the sum of any number of integers has for its absolute value the difference between the sum of the positive integers and the sum of the opposites of the negative integers; and for its sign, the sign of those integers that gave the larger sum.*

45. *Algebraic addition is thus both commutative and associative:* commutative, because commutation is but a special case of change of order; associative, because by successive changes of order as we add we can get any desired grouping.

Let the student compare §9 and show how, by changes of order, to get

$$[(+a)+(\overline{-b+(-c)}]+\{(+d)+[(-e)+\overline{(-f)+(+g)}]\};$$

from $(+a)+(-b)+(-c)+(+d)+(-e)+(-f)+(+g)$.

46. From the associative law it follows that *the sum of two or more addition and subtraction expressions is the sum of the numbers entering into the expressions.*

E.g., $(-a+b-c-d)+(-e+k-f-g)$
$= -a+b-c-d-e+k-f-g.$

47. *If all the numbers in one addition and subtraction expression are the opposites of the numbers in another addition and subtraction expression, the expressions are called opposites of each other.* Plainly, their values, i.e. the single numbers to which they are reducible, are opposite. But to subtract a number is to add its opposite; therefore, *to subtract an addition and subtraction expression is to add its opposite.*

E.g., $(a-b+c-d-e)-(-f-g+k-l)$
$= a-b+c-d-e+f+g-k+l.$

48. The expressions considered may themselves be sums of expressions and these in turn sums of others, and so on.

E.g., $$a-\{b+[c+(f-\overline{g-k})]\}.$$

This means that the number $g-k$ is to be taken from f, the result added to c, that result added to b, and this last result subtracted from a.

By §§46, 47,

$$a-\{b+[c+(f-\overline{g-k})]\}=a-b-[c+(f-\overline{g-k})]$$
$$=a-b-c-(f-\overline{g-k})=a-b-c-f+\overline{g-k}$$
$$=a-b-c-f+g-k.$$

Of course, we could have just as well removed the signs of inclusion, beginning with the inmost and proceeding outwards.

But the final result can be written without going through the intermediate steps. For notice: each number is acted upon not only by the sign immediately before it, but also by the sign of each and every expression in which it is included.

Thus k is to be counted backward from g, the reverse of backward, i.e. forward, from f, forward from c, forward from b, backward from a. Hence its final sign is $-$.

Similarly, we can get the signs of all of the other numbers.

In general, *every minus sign acting upon a number reverses the number, and hence the sign of the number is finally* $+$ *or* $-$ *according as an even or an odd number of minus signs act upon the number.*

The student will do well to build up complicated addition and subtraction expressions and simplify them by the foregoing rule. He can test his work by substituting actual numbers for the letters in both the original expression and the final simplification. The calculated value of each should be the same.

49. Evidently any addition and subtraction expression is by the above processes reducible to a positive or a negative integer; and we cannot, therefore, by addition and subtraction of integers get aught save integers.

50. Let a, b, and d be positive integers, and let $a+b=c$.

Then $$cd=(a+b)d=ad+bd,$$

and $$ad=cd-bd;$$

but $$a=c-b, \quad \text{and} \quad ad=(c-b)d;$$

$$\therefore (c-b)d=cd-bd.$$

In words, *multiplication is distributive with regard to subtraction if the result of the subtraction is positive.*

We have not defined multiplication when negative numbers enter. Let us assume it such that this law still holds. Then, just as $(c-b)d=cd-bd$, we have

$$(b-c)d=bd-cd, \text{ or } (-a)d=-ad.$$

The product of a negative number by a positive is a negative number; and by commutation, a positive number multiplied by a negative likewise gives a negative result.

Further,

$$(-a)(-d) = (b-c)(-d) = b(-d) - c(-d)$$
$$= -bd + cd = cd - bd = ad.$$

A negative number multiplied by a negative gives a positive number.

It goes without saying that, in all cases, the absolute value of the product is the product of the absolute values of the factors.

51. In like manner, *the continued product of any number of positive and of negative integers has for its absolute value the product of the absolute values of the integers, and its sign will be* $+$ *or* $-$ *according as the number of negative integers is even or odd.*

52. Since the absolute value of the product is the same when negatives enter as when they do not, and since the sign of the product depends only upon the number of negatives entering, it follows that the multiplication can be performed in any order. In other words, the commutative and associative laws still hold.

53. Let $ad = c$; where a, d, and c are positive integers.

Then $$(-a)d = a(-d) = -ad = -c,$$

and $$(-a)(-d) = ad = c.$$

But division is the undoer of multiplication:

$$\therefore \quad (-c) \div d = c \div -d = -a = -(c \div d),$$

and $$(-c) \div -d = a = c \div d;$$

or $$\frac{c}{-d} = \frac{-c}{d} = -\frac{c}{d}, \quad \text{and} \quad \frac{-c}{-d} = \frac{c}{d}.$$

These equations, true when division is exact, we assume true when division is inexact, thus defining the negative fractions to which reference has been made.

With fractions as with integers, opposites have the same absolute values but opposite signs.

We say that two negative fractions are equal if the opposite positive fractions are equal;

$$-\frac{a}{b} = -\frac{c}{d}, \text{ if } \frac{a}{b} = \frac{c}{d}.$$

Hence $\frac{a}{b} = \frac{c}{d}$ involves—

$$\frac{-a}{-b} = \frac{-c}{-d}, \quad \frac{-a}{-b} = \frac{c}{d}, \quad \frac{a}{b} = \frac{-c}{-d}, \quad \frac{-a}{b} = \frac{-c}{d},$$

$$\frac{-a}{b} = \frac{c}{-d}, \quad \frac{a}{-b} = \frac{-c}{d}, \quad \frac{a}{-b} = \frac{c}{-d}.$$

Notice that just as $\frac{a}{b} = \frac{c}{d}$ requires $ad = bc$, so in all the other fraction equalities the cross-products are equal.

54. We know that $a > b$ involves $-a < -b$. We now agree that $\frac{a}{b} > \frac{c}{d}$ shall involve $-\frac{a}{b} < -\frac{c}{d}$. But the necessary and sufficient condition for $\frac{a}{b} \gtreqless \frac{c}{d}$ is $ad \gtreqless bc$. The corresponding condition, therefore, with negative fractions is

$$-\frac{a}{b} \lesseqgtr -\frac{c}{d}, \quad \text{if} \quad -ad \lesseqgtr -bc.$$

It easily follows that if a fraction lies between two others its opposite lies between the opposites of the two others. Consequently between any two negative fractions there lie an infinite number of other negative fractions.

55. We return to *positive* fractions. In § 37 we saw that if $a \div b$ was an integer, then to multiply a by any number or to

divide b by any factor of b would multiply $a \div b$ by that number or factor. E.g.,

$$24 \div 6 = 4, \quad \text{and} \quad (24 \times 2) \div 6 = 24 \div (6 \div 2) = 8 = 4 \times 2.$$

?

We assume this true of all division expressions, i.e. of all fractions.

Thus, no matter what positive integers a, b and c are,

$$\frac{a}{b} \times c = \frac{ac}{b} = \frac{a}{b \div c};$$

and, if the commutative law is to hold, Contradicts § 31

$$c \times \frac{a}{b} = \frac{ca}{b} = c \times a \div b.$$

In words, *to multiply by a fraction means to multiply by the numerator and then divide by the denominator.*

Since not only $c \times \frac{a}{b}$ but also $\frac{c}{b} \times a$ gives $\frac{ca}{b}$, it follows that $c \times \frac{a}{b} = c \div b \times a$, and so we can effect multiplication by a fraction by first dividing by the numerator and then multiplying by the denominator.

In particular, $\frac{a}{b}$ is either $1 \times a \div b$ or $1 \div b \times a$, is either a part of a multiple of unity or a multiple of a part of unity.

56. Immediately from the definition of multiplication by a fraction, we get

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd};$$

and *the product of any number of fractions* is the fraction $\dfrac{\textit{product of numerators}}{\textit{product of denominators}}$.

Plainly, the operation is both commutative and associative.

57. Because division undoes multiplication, *to divide by a fraction is to multiply by the denominator and then divide by the numerator.*

Thus, to divide by $\frac{a}{b}$ is to multiply by b and then divide by a; but this is the same as to multiply by $\frac{b}{a}$.

Now $\frac{a}{b} \times \frac{b}{a} = \frac{ab}{ab} = 1$. Two such numbers whose product is positive unity we call reciprocals of each other. For instance, the multiples of unity 2, 3, 4, 5, . . . , are reciprocals of the parts of unity $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$, . . .

We can now say, *to* $\left\{\begin{matrix} \textit{multiply} \\ \textit{divide} \end{matrix}\right\}$ *by a number is to* $\left\{\begin{matrix} \textit{divide} \\ \textit{multiply} \end{matrix}\right\}$ *by the reciprocal number.*

58. Having a chain of multiplications and divisions to perform, we can turn the divisions into multiplications-by-reciprocals; § 55 then tells us that the operations can be performed in any order. They could therefore previous to the change of the divisions into multiplications. The student may give examples.

59. Consider the expression

$$a \div \{b \times [c \times (f \div \overline{g \div k})]\}.$$

Here $g \div k = \frac{g}{k}$; $f \div \overline{g \div k} = f \div \frac{g}{k} = f \times \frac{k}{g} = \frac{fk}{g}$;

$$c \times (f \div \overline{g \div k}) = \frac{cfk}{g}; \quad b \times [c \times (f \div \overline{g \div k})] = \frac{bcfk}{g};$$

and finally,

$$a \div \{b \times [c \div (f \div \overline{g \div k})]\} = a \div \frac{bcfk}{g} = \frac{ag}{bcfk}$$

$$= a \div b \div c \div f \times g \div k.$$

This result we could have foreseen by noticing that k is a divisor of g, a divisor of f's divisor or a multiplier of f, a multiplier of c, a multiplier of b; and since b is a divisor of a, k is thus a divisor of a. And any number in the expression is a multiplier or a divisor of a according as it is affected by an even or an odd number of division signs.

Of course our manner of indicating division has nothing to do with this result.

Thus, in $$\cfrac{\cfrac{a}{b \div \cfrac{c}{m}}}{k \div \cfrac{d}{f \div \cfrac{r}{s}}},$$ s divides r, multiplies f, divides d, multiplies k, divides m, multiplies c, divides b, multiplies a. Likewise r divides, f multiplies, d divides, k multiplies, m divides, c multiplies, and b divides a.

$$\therefore \text{ the expression is } \frac{ackfs}{bmdr}.$$

The student should practise himself in reductions similar to the above until he can perform them with ease and certainty. Let him compare § 48.

60. The method of reduction just exhibited can plainly be applied to all multiplication and division expressions. Plainly also, any chain of multiplications and divisions is indicated by some such expression. We cannot, therefore, by the multiplication and division of positive integers and fractions get aught save positive integers and fractions.

61. The addition of fractions has not yet been defined. We assume it to be such a process that multiplication is distributive with reference to it. Then, if $\frac{a}{b}$ and $\frac{c}{d}$ are two positive fractions, we have by this convention

$$\left(\frac{a}{b} + \frac{c}{d}\right) \times bd = ad + bc;$$

and thence $$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}.$$

In the same way,

$$\frac{a_1}{b_1}+\frac{a_2}{b_2}+\frac{a_3}{b_3}+\ldots+\frac{a_n}{b_n}$$
$$=\frac{a_1b_2b_3\ldots b_n+b_1a_2b_3\ldots b_n+\ldots+b_1b_2\ldots b_{n-1}a_n}{b_1b_2b_3\ldots b_n}.$$

Thus, *the sum of any number of fractions is the fraction whose numerator is the sum of the products gotten by multiplying the numerator of each given fraction by the denominators of all the other given fractions, and whose denominator is the product of all the given denominators.*

Evidently the commutative and associative laws hold as with integers.

62. Just as $a+b=c$ gave $c-b=a$, so $\frac{a}{b}+\frac{c}{d}=\frac{e}{f}$ gives $\frac{e}{f}-\frac{c}{d}=\frac{a}{b}$. Thus subtraction of fractions enters as did subtraction of integers, then negative fractions, and algebraic addition and subtraction of fractions. Accordingly, the sum of $\frac{a}{b}$, $-\frac{c}{d}$, $-\frac{e}{f}$, and $\frac{g}{h}$ is

$$\frac{a}{b}-\frac{c}{d}-\frac{e}{f}+\frac{g}{h}=\frac{adfh-bcfh-bdeh+bdfg}{bdfh}.$$

The student may show that the definition of a negative fraction just suggested agrees with that of § 53.

63. We can now in §§ 50, 51, 52, 53 remove the restriction, express or implied throughout, that the letters should stand for positive integers, and allow them to stand for fractions as well. Thus we easily show that *the absolute value of any multiplication and division expression is the same as if all the numbers entering therein were positive, while the sign is positive or negative according as there are an even or an odd number of minus signs affecting the factors, direct or reciprocal, of the expression.*

Here, by a *direct factor* we mean one that multiplies the final value of the expression; by a *reciprocal factor*, one that divides the final value of the expression.

If in any multiplication and division expression we change all the factors into their reciprocals, the resulting expression is the reciprocal of the original expression, and to multiply by either expression is the same as to divide by the other. The student may give examples. Compare § 47.

Can two expressions be both opposites and reciprocals?

If I change all the factors of a multiplication and division expression into their opposites, will the resulting expression be the opposite of the original expression?

64. Any expression that can be built up by additions, subtractions, multiplications, and divisions, let the grouping be ever so intricate and involved, can, by the mere performance of the indicated operations, be reduced to a simple positive or negative fraction. Furthermore, all the rules applying to the addition and subtraction, the multiplication and division of positive integers, apply to these more complicated expressions; for they apply to the equivalents of these expressions, the simple fractions.

65. Raising of negatives and fractions to positive integral powers requires no explanation. Evidently, $\left\{\begin{matrix}\textit{even}\\ \textit{odd}\end{matrix}\right\}$ *powers of negatives are* $\left\{\begin{matrix}\textit{positive}\\ \textit{negative}\end{matrix}\right\}$; *and any power of a fraction = the new fraction* $\dfrac{\textit{power of numerator}}{\textit{power of denominator}}$.

Likewise any multiplication and division expression is powered by a positive integer when all the factors, direct and reciprocal, of the expression are so powered. In other words, we power the expression by distributing the index of the power over the factors of the expression.

66. Since evolution and taking logarithms are the inverses of involution,

$$\left(\frac{a}{b}\right)^c = \frac{d}{e} \text{ requires both } \sqrt[c]{\frac{d}{e}} = \frac{a}{b}, \text{ and } \log_{\frac{a}{b}} \frac{c}{d} = c;$$

where $\frac{a}{b}$ and $\frac{c}{d}$ must agree in sign if c is odd.

67. Consider the expression $a^b \div a^c$, where b and c are positive integers. By our definitions the expression is

$$(a \times a \times a \times a \times \ \ldots \text{ to } b \ a\text{'s}) \div (a \times a \times a \times a \times \ \ldots \ c \ a\text{'s}).$$

Assuming $b > c$ reduces the above to

$$1 \times a \times a \times a \times a \times \ \ldots \text{ to } (b - c) \ a\text{'s} = a^{b-c};$$

while $b = c$ gives unity, and $b < c$ gives

$$1 \div a \div a \div a \div a \div \ \ldots \text{ to } (c - b) \ a\text{'s}.$$

The agreement that the last two results as well as the first shall be denoted by a^{b-c} extends our notion of powering, and gives us these definitions.

Powering by a positive index is repeatedly multiplying unity by the base.

Powering by zero is leaving unity alone.

Powering by a negative index is repeatedly dividing unity by the base.

In all cases the absolute value of the index is the number of times that the base operates upon unity.

Thus, $a^{-b} = \dfrac{1}{a^b}$, $\left(\dfrac{1}{a}\right)^{-b} = a^b$, $(2^3 . 3^2 . 4^{-2})^{-2} \div (2^{-1} . 3^2 . 4)^{-3} = 2^{-3} . 3^2 . 4^7 = 2^{11} . 3^2 = 18432.$

Express as a simple fraction without negative indices

$$\left(\frac{a^{-2} \div a^3 b^{-4}}{a^{-3} b^7 \div a^4 b^{-6}}\right)^{-2} \times \left(\frac{a^2 b^{-1}}{a^{-3}}\right)^{-3} \div \left(\frac{b^{-3}}{a^2}\right)^{-4}.$$

Also, write an equivalent expression without denominators or the signs $\div$.

68. Before considering the meaning of fractional indices, the student should prove these equalities. We assume that the indicated roots can always be taken. Thus the second equality below is to be understood: "If there is a qth root of a and a pth root of that qth root, and if there is also a qth root

of a and a pth root of that qth root, then the pth root of the qth root is the same as the qth root of the pth root, and each is the pqth root of a."

$$\sqrt[p]{a^q} = (\sqrt[p]{a})^q = \sqrt[kp]{a^{kq}}; \quad \sqrt[p]{\sqrt[q]{a}} = \sqrt[q]{\sqrt[p]{a}} = \sqrt[pq]{a}; \quad \sqrt[p]{a} \,.\, \sqrt[p]{b} = \sqrt[p]{ab};$$

$$\frac{\sqrt[p]{a}}{\sqrt[p]{b}} = \sqrt[p]{\frac{a}{b}}; \quad \sqrt[p]{a} \,.\, \sqrt[q]{b} = \sqrt[pq]{a^q b^p}; \quad \frac{\sqrt[p]{a^q}}{\sqrt[p]{b^q}} = \sqrt[p]{\left(\frac{a}{b}\right)^q}; \quad \frac{\sqrt[p]{a}}{\sqrt[q]{b}} = \sqrt[pq]{\frac{a^q}{b^p}};$$

$$\sqrt[p]{a} \div \sqrt[q]{a} = \sqrt[pq]{a^{q-p}}; \quad \sqrt[p]{a} \cdot \sqrt[q]{a} = \sqrt[pq]{a^{p+q}}.$$

The product of the pth root, the qth root, and the rth root of a is what root of what power of a?

The pth root of a to the rth power, multiplied by the qth root of a to the s power, is what root of what power of a?

To define powering by a fraction, suppose that $(a^b)^c = a^{bc}$ holds just the same when b is a fraction as when it is an integer.

Then $$\left(a^{\frac{b}{c}}\right)^c = a^b, \quad \text{and so} \quad a^{\frac{b}{c}} = \sqrt[c]{a^b}.$$

In particular, $$a^{\frac{1}{c}} = \sqrt[c]{a}.$$

Let the student now express all of the foregoing equalities in the new notation. He can then prove that $a^{\frac{b}{c}} \times a^{\frac{d}{e}} = a^{\frac{b}{c}+\frac{d}{e}}$, and that $a^{\frac{b}{c}} \times d^{\frac{b}{c}} = (ad)^{\frac{b}{c}}$.

69. If $a^{\frac{b}{c}} = d$, then $d^{\frac{c}{b}} = a$. But, if we use a fractional root-index, $a^{\frac{b}{c}} = d$ gives $\sqrt[\frac{b}{c}]{d} = a$; and so $d^{\frac{c}{b}} = \sqrt[\frac{b}{c}]{d}$.

Of course, if $\left(\frac{a}{b}\right)^{\frac{c}{d}} = \frac{e}{f}$, $\log_{\frac{a}{b}} \frac{e}{f} = \frac{c}{d}$. Find x below.

$$\text{Log}_{\frac{2}{3}} \tfrac{27}{8} = x, \quad \log_{\frac{1}{4}} x = 32, \quad \log_x 64 = \tfrac{3}{2}.$$

IV. Incommensurables.

70. Evolution introduces new expression-numbers. For example, $\sqrt{2}$ is such a number. Obviously, the square of no integer is 2; for $(\pm 1)^2 = 1$, $(\pm 2)^2 = 4$, $(\pm 3)^2 = 9, \ldots,$ numbers all different from 2. Neither is there a fraction $\frac{a}{b}$ such that $\left(\frac{a}{b}\right)^2 = 2$.

For this requires $a^2 = 2b^2$.

Is a odd and $= 2k + 1$, say?

Then $a^2 = (2k + 1)(2k + 1) = 4k^2 + 4k + 1 =$ an odd number.

But $2b^2$ is an even number.

$$\therefore a^2 \neq 2b^2 \text{ unless } a \text{ is even.}$$

Suppose, then, a is even and $= 2c$.

It follows that $a^2 = 4c^2 = 2b^2$, and so $b^2 = 2c^2$.

This requires b even, say $b = 2d$; and so $c^2 = 2d^2$.

Just as, at the start, a was shown to be even, we can now show c to be even. But c is the result of dividing a by 2. Consequently $a \div 2$ is even; and going on, $c \div 2$ or $a \div 2 \div 2$ is even, and likewise $a \div 2 \div 2 \div 2$, $a \div 2 \div 2 \div 2 \div 2, \ldots$ But every division gives a smaller number, and so exact division must cease either when we come to the smallest integer 1, or before that; that is, some time or other we get the square of an odd number equal to an even number, a proved impossibility. Therefore $\sqrt{2}$ is not a fraction.

In like manner it can be shown that $\sqrt{3} = \frac{a}{b}$ requires

$$p^2 = 3q^2, \text{ where } p = 3l \pm 1;$$

i.e., $$3lp \pm p = 3q^2, \text{ or } 3(lp \pm p) \mp 1 = 3q^2,$$

a manifest absurdity, l, p, and q being integers.

Let the student show also that $\sqrt[3]{2}$, $\sqrt[4]{2}$, ... $\sqrt[n]{2}$, are neither integers nor fractions.

Later he can show that all powers of fractions are fractions, and that consequently if an integer has no integral root it can have no fractional one.

71. The new numbers are *surds* or *irrationals* and belong to the large class of numbers called *incommensurables because not having a common measure with unity.*

To define the equality and inequality of surds, consider how the equality and inequality of expressions has so far been tested.

We had $$a - b \gtreqless c - d, \text{ if } a + d \gtreqless c + b,$$

and $$a \div b \gtreqless c \div d, \text{ if } a \times d \gtreqless c \times b;$$

where of course a, b, c, and d are supposed positive integers.

We ought to expect, then, using the p and up notation of

§ 17, $$a \text{ up } b \gtreqless c \text{ up } d, \text{ if } a \text{ p } d \gtreqless c \text{ p } b;$$

or, in ordinary notation,

$$\sqrt[b]{a} \gtreqless \sqrt[d]{c}, \text{ if } a^d \gtreqless c^b.$$

When we can actually take the bth root of a and the dth root of c, the condition certainly applies: for suppose $\sqrt[b]{a} = k$, and $\sqrt[d]{c} = l$; then $a^d = k^{bd}$ and $c^b = l^{db}$, and plainly

$$k^{bd} \gtreqless l^{bd}, \text{ if } k \gtreqless l,$$

which is precisely our condition. The condition is fulfilled, moreover, even when, in place of a and c, we write fractions $\frac{a}{e}$ and $\frac{c}{f}$. If we now make this test universal we shall but be

following our previous methods. Thus, for surds as well as exact roots, we write

$$\sqrt[b]{\frac{a}{e}} \begin{smallmatrix}>\\=\\<\end{smallmatrix} \sqrt[d]{\frac{c}{f}}, \text{ if } \left(\frac{a}{e}\right)^d \begin{smallmatrix}>\\=\\<\end{smallmatrix} \left(\frac{c}{f}\right)^b.$$

If $b = d$, this becomes

$$\sqrt[b]{\frac{a}{e}} \begin{smallmatrix}>\\=\\<\end{smallmatrix} \sqrt[b]{\frac{c}{f}}, \text{ if } \frac{a}{e} \begin{smallmatrix}>\\=\\<\end{smallmatrix} \frac{c}{f}.$$

Thus, we say

$$\sqrt{2} > 1, \quad 1.4, \quad 1.41, \quad 1.414, \quad 1.4142,$$

and $$\sqrt{2} < 2, \quad 1.5, \quad 1.42, \quad 1.415, \quad 1.4143;$$

because the squares of the first set of numbers,

$$1, \quad 1.96, \quad 1.9881, \quad 1.999396, \quad 1.99996164,$$

are all less than 2: while the squares of the second set,

$$4, \quad 2.25, \quad 2.0164, \quad 2.002225, \quad 2.00024449,$$

are all greater than 2.

$\frac{a}{e}$ and $\frac{c}{f}$ above may themselves be powers of fractions. Say they are $\left(\frac{k}{l}\right)^m$ and $\left(\frac{n}{o}\right)^p$. This gives

$$\sqrt[b]{\left(\frac{k}{l}\right)^m} \begin{smallmatrix}>\\=\\<\end{smallmatrix} \sqrt[d]{\left(\frac{n}{o}\right)^p}, \text{ if } \left(\frac{k}{l}\right)^{md} \begin{smallmatrix}>\\=\\<\end{smallmatrix} \left(\frac{n}{o}\right)^{pb};$$

or, $$\left(\frac{k}{l}\right)^{\frac{m}{b}} \begin{smallmatrix}>\\=\\<\end{smallmatrix} \left(\frac{n}{o}\right)^{\frac{p}{d}}, \text{ if } k^{md}o^{pb} \begin{smallmatrix}>\\=\\<\end{smallmatrix} n^{pb}l^{md}.$$

Arrange the expressions in the following groups in the order of their magnitudes:

$$\sqrt[5]{\tfrac{2}{3}},\quad \sqrt[4]{\tfrac{1}{2}},\quad \sqrt[3]{\tfrac{4}{7}};\quad \sqrt[5]{\tfrac{3}{2}},\quad \sqrt[4]{2},\quad \sqrt[3]{\tfrac{7}{4}};$$

$$(\tfrac{1}{2})^{\frac{3}{7}},\quad (\tfrac{2}{3})^{\frac{4}{9}},\quad (\tfrac{5}{4})^{\frac{2}{3}};\quad \sqrt[3]{4},\quad \sqrt[4]{5},\quad \sqrt[5]{6}.$$

Prove that $\left(\frac{a}{b}\right)^{\frac{c}{d}} > 1$, if $a > b$ and $c > d$; and that otherwise $\left(\frac{a}{b}\right)^{\frac{c}{d}} \overline{\overline{<}}\ 1$. What conditions give $\left(\frac{a}{b}\right)^{\frac{c}{d}} = 1$? How can $a^{a^k} = a^{ka}$? Show that $(\frac{9}{4})^{\frac{27}{8}} = (\frac{27}{8})^{\frac{9}{4}}$.

72. We saw in § 42 that between any two positive fractions were an infinite number of other fractions. Hence between any two squares of fractions are an infinity of other squares of fractions, between any two cubes an infinity of other cubes, and so on. It follows that there are always two fractions as close together as we please between which any required root of positive integer or fraction must lie.

A root lying between positives we naturally call positive, and we define its opposite, negative of course, as the number lying between the opposites of the including positives.

In the same way, we say that the reciprocal of a surd is the number lying between the reciprocals of the including numbers.

73. Taking logarithms leads to incommensurables. Thus there is no integer or fraction $\log_2 3$.

For, suppose $$\log_2 3 = \frac{a}{b};$$

then $$2^a = 3^b.$$

Now 2 divides 2^a, and therefore it divides 3^b or $3 \cdot 3^{b-1}$. This makes it divide 3^{b-1}; for otherwise there would be a remainder of 1; and in the three 3^{b-1}'s a remainder of 3, which 2 does not divide. In like manner, 2 should divide 3^{b-2}, 3^{b-3}, . . . , 3^4, 3^3, 3^2, 3, an absurdity.

Consequently 2 does not divide 3^b and $2^a \neq 3^b$, nor is there a fraction $\frac{a}{b}$ such that $\log_2 3 = \frac{a}{b}$.

Again, $\log_{10} 2$, $\log_{10} 3$, $\log_{10} 4$, $\log_{10} 6$, . . . , are incommensurables. For all powers of 10 end in a cipher, but no powers of 2, 3, 4, 5, or 6 do.

74. Whether or no these incommensurables can be expressed as surds need not now concern us. We can treat them as we did the surds. Thus, if $2^{\frac{a}{b}} > 3$ and $2^{\frac{c}{d}} < 3$, we say that $\frac{a}{b} > \log_2 3 > \frac{c}{d}$.

More generally,

$$\text{if} \quad \left(\frac{a}{b}\right)^{\frac{c}{d}} > \frac{e}{f} \quad \text{and} \quad \left(\frac{a}{b}\right)^{\frac{g}{h}} < \frac{e}{f},$$

$$\text{then} \quad \log_{\frac{a}{b}} \frac{e}{f} \text{ lies between } \frac{c}{d} \text{ and } \frac{g}{h}.$$

$$\text{When} \quad \frac{a}{b} > \text{unity}, \ \frac{c}{d} > \log_{\frac{a}{b}} \frac{e}{f} > \frac{g}{h}.$$

$$\text{When} \quad \frac{a}{b} < \text{unity}, \ \frac{c}{d} < \log_{\frac{a}{b}} \frac{e}{f} < \frac{g}{h}.$$

Since between any two fractions $\frac{c}{d}$ and $\frac{g}{h}$ there are an infinite number of other fractions as close together as you please; and since if a fraction lies between two others the result of powering by that fraction will lie between the results of powering by the including fractions; it follows that $\log_{\frac{a}{b}} \frac{e}{f}$ is hemmed in as closely as you please.

Let the student put the above conditions in the b and ub notation of § 17.

75. Notice that for both sorts of incommensurables we have all fractions divided into two sets, such that all in one set are

less than any in the other set, and so that, moreover, there is no largest fraction in the set of smaller ones, nor smallest fraction in the set of larger ones.

Were there a largest fraction $\frac{a}{b}$ in the set of smaller ones, then no fraction could be ever so little larger than it without falling into the set of larger ones, and the incommensurable α hemmed in between the two sets could not be ever so little larger nor the least bit smaller than $\frac{a}{b}$. In other words, the incommensurable is the commensurable $\frac{a}{b}$, a palpable contradiction.

By this property we define incommensurables.

An incommensurable, we say, *is a number that divides all fractions into two sets A and B such that any fraction* $\frac{a}{a'}$ *from A is less than any fraction* $\frac{b}{b'}$ *from B, but yet no* $\frac{a}{a'}$ *is largest, nor any* $\frac{b}{b'}$ *smallest.*

These incommensurables we denote by Greek letters, and agree, if α is an incommensurable and $\frac{a}{a'}$, $\frac{b}{b'}$ any fraction whatever from the two sets of limiting fractions, that

$$\frac{a}{a'} < \alpha < \frac{b}{b'}.$$

It follows immediately that *the opposites and reciprocals of incommensurables,* as defined (§ 74), *are themselves incommensurable.*

76. *Two incommensurables α and β are called equal if every fraction smaller than α is also smaller than β and every fraction larger than α is also larger than β: are equal, in brief, if their inclusives are equal.* Notice that the same definition applies to commensurables.

77. *If two incommensurables α and β are connected by addition, multiplication, or powering, we agree that the results shall be hemmed in by the results of the same operations on the inclusives of α and β.*

$$\text{Thus, if } 0 < \frac{a}{b} < \alpha < \frac{c}{d}, \text{ and } 0 < \frac{e}{f} < \beta < \frac{g}{h},$$

$$\frac{a}{b} + \frac{e}{f} < \alpha + \beta < \frac{c}{d} + \frac{g}{h}, \quad \frac{a}{b} \times \frac{e}{f} < \alpha \times \beta < \frac{c}{d} \times \frac{g}{h},$$

while if $\frac{a}{b} > 1$, $\left(\frac{a}{b}\right)^{\frac{e}{f}} < \alpha^{\beta} < \left(\frac{c}{d}\right)^{\frac{g}{h}}$, taking positive roots only.

Of course the inequalities are supposed to hold if in place of either α or β we have a commensurable, and everybody knows that they must hold when in place of both α and β we write commensurables.

Under the same conditions as above write the inclusives for $\alpha - \beta$, $\alpha \times - \beta$, and $\alpha^{-\beta}$.

Show that the sum of a commensurable and an incommensurable is incommensurable, as is likewise the product.

Show that the sum of two incommensurables may be commensurable, as may likewise the product.

Show that the primary operations with incommensurables are uniform, and that the commutative, associative, distributive, and index laws hold as with commensurables.

The last problem is especially simple. For example,

$$\frac{a}{b} + \frac{e}{f} < \alpha + \beta < \frac{e}{f} + \frac{g}{h} \text{ requires } \frac{e}{f} + \frac{a}{b} < \beta + \alpha < \frac{g}{h} + \frac{c}{d}.$$

78. Since subtraction is the addition of an opposite; division, multiplication by a reciprocal; and evolution, raising to a reciprocal power,—we need not specially consider these processes.

As for taking logarithms, $\log_a \beta = \gamma$ if $\alpha^\gamma = \beta$. More explicity, if $\alpha^{\frac{a}{b}} < a^\gamma < \alpha^{\frac{c}{d}}$, then γ lies between $\frac{a}{b}$ and $\frac{c}{d}$.

The student may show that

$$\log_a \beta\gamma = \log_a \beta + \log_a \gamma;\ \log_a \frac{\beta}{\gamma} = \log_a \beta - \log_a \gamma;$$

$$\log_a \beta^\gamma = \gamma \log_a \beta;\ \log_a \sqrt[n]{\beta} = \frac{1}{n} \log_a \beta;$$

$$\log_a \beta \log_\beta \gamma = \log_a \gamma;\ \ \log_a \beta \log_\gamma \delta = \log_a \delta \log_\gamma \beta = \log_{a\gamma} \beta\delta;$$

$$\log_a \beta = 1 \div \log_\beta \alpha;\ \alpha^{\log_\beta \gamma} = \gamma^{\log_\beta a};\ \alpha^{\log_a \gamma} = \gamma;$$

$$\log_{\sqrt{2}} 8\sqrt{2} = ?\ \log_{2\sqrt{3}} 12 = ?\ \log_{1+\sqrt{2}} (3 + 2\sqrt{2}) = ?$$

$$\log_{a\sqrt{b}\,.\,\sqrt[3]{c}}\, a^7 b^3 c^2 \sqrt{b}\,.\, \sqrt[3]{c} = ?\ \log_{\frac{2}{3}}\,.\,\frac{2\sqrt{2}}{3\sqrt{3}} = ?$$

79. By definition, $\frac{a}{b} < \sqrt{2} < \frac{c}{d}$ if $\frac{a^2}{b^2} < 2 < \frac{c^2}{d^2}$. But also, by definition, $\frac{a}{b} < \sqrt{2} < \frac{c}{d}$ requires $\frac{a^2}{b^2} < (\sqrt{2})^2 < \frac{c^2}{d^2}$. Consequently $(\sqrt{2})^2 = 2$. Similarly, $(\sqrt[n]{a})^n = a = \sqrt[n]{a^n}$ whether $\sqrt[n]{a}$ be commensurable or incommensurable.

As between any two fractions there are an infinite number of fractions, so between any two exact integral nth powers of two fractions there are an infinite number of nth powers of fractions infinitely close together. These can be used to hem in incommensurables without the help of fractions not perfect nth powers. In fact, howsoever close together the fractions $\frac{a}{b}$ and $\frac{c}{d}$ enclosing the incommensurable α may be taken $\left(\frac{a}{b} < \alpha < \frac{c}{d}\right)$, there are an infinite number of fractions $\frac{e}{f}$ such that $\frac{a}{b} < \frac{e^n}{f^n} < \frac{c}{d}$.

These fractions are those from the inclusives of $\sqrt[n]{\frac{a}{b}}$ and $\sqrt[n]{\frac{c}{d}}$ that are greater than $\sqrt[n]{\frac{a}{b}}$ and less than $\sqrt[n]{\frac{c}{d}}$. Of these an infinite number have their nth powers less than α and an infinite number their nth powers greater than α; for otherwise α would fall in either with one of the nth powers or else with $\frac{a}{b}$ or $\frac{c}{d}$, and could not be incommensurable.

It is tacitly assumed above that $\sqrt[n]{\frac{a}{b}}$ and $\sqrt[n]{\frac{c}{d}}$ are incommensurable. All that would happen, if they were not, is that one or both of them would fall in with nth powers of the fractions $\frac{e}{f}$, and the reasoning would not be invalidated.

In like manner, *any series of fractions that are not limited in size and between any two of which, howsoever close together, there lies a fraction and so an infinity of fractions of the series, may be used to hem in incommensurables.*

In particular, we may use decimal fractions.

The condition "not limited in size" is important. If, for instance, no fractions of the series were larger than 10, the series could not be used to hem in an incommensurable larger than 10. If, again, all fractions of the series were either larger than 10 or smaller than 5, no incommensurable between 5 and 10 could be hemmed in by the series.

An interesting application of the above principles is afforded in the formal proof that $\sqrt[m]{p^n} = (\sqrt[m]{p})^n$. Let $\sqrt[m]{p^n} = \alpha$ and $\sqrt[m]{p} = \beta$; we are to prove that $\alpha = \beta^n$.

We have $\frac{a}{b} < \alpha < \frac{c}{d}$, if $\frac{a^m}{b^m} < p^n < \frac{c^m}{d^m}$;

and $\frac{e}{f} < \beta < \frac{g}{h}$, if $\frac{e^m}{f^m} < p < \frac{g^m}{h^m}$.

Now $\qquad \frac{e}{f} < \beta < \frac{g}{h} \quad$ involves $\quad \frac{e^n}{f^n} < \beta^n < \frac{g^n}{h^n}$;

while $\qquad \frac{e^m}{f^m} < p < \frac{g^m}{h^m} \quad$ involves $\quad \frac{e^{mn}}{f^{mn}} < p^n < \frac{g^{mn}}{f^{mn}}$

But $\qquad \frac{e^{mn}}{f^{mn}}$ and $\frac{g^{mn}}{h^{mn}}$ are mth powers of $\frac{e^n}{f^n}$ and $\frac{g^n}{h^n}$.

Consequently β^n as well as α lies between numbers whose mth powers hem in p^n. The numbers $\frac{e^n}{f^n}$ are thus among the numbers $\frac{a}{b}$, and the numbers $\frac{g^n}{f^n}$ among the numbers $\frac{c}{d}$, and $\alpha = \beta^n$.

80. Consider the expression $\frac{\alpha}{\beta}$, where α and β are two positive incommensurables. Let $0 < a < \alpha < a'$ and $0 < b < \beta < b'$, where the new letters may be fractions. Then

$$\frac{a}{b'} < \frac{\alpha}{\beta} < \frac{a'}{b}.$$

Now $\frac{\alpha}{\beta}$ may be either integral, fractional, or incommensurable. Thus, if $\alpha = 2\beta$, $a < 2b'$ and $a' > 2b$, so that $\frac{a}{b'} < 2 < \frac{a'}{b}$. Again, if $\alpha = \frac{2}{3}\beta$, $\frac{a}{b'} < \frac{2}{3} < \frac{a'}{b}$; while if $\alpha = \beta\sqrt{2}$, $\frac{a}{b'} < \sqrt{2} < \frac{a'}{b}$.

Remove the restriction that α and β shall be positive, and $\frac{\alpha}{\beta}$ stands for all the sorts of numbers with which we have had to do. For this expression we have a name. We call it a *ratio* and define it as *the number*, be it positive, negative, integral, fractional, or incommensurable, *by which we must multiply one number to get another.* The ratio $\frac{\alpha}{\beta}$ of α to β is

frequently denoted by $\alpha : \beta$, where the sign : takes precedence of all other symbols of operation.

$$\text{E.g.,}\qquad \alpha + \beta \times \gamma : \delta \div \epsilon - \zeta = \frac{\alpha + \beta\gamma}{\frac{\delta}{\epsilon} - \zeta}.$$

There is another unique convention about the use of the ratio sign.

We write $\qquad a : b : c : d = k : l : m : n$

to mean that the ratio of any two numbers on the left is the same as the ratio of the corresponding two on the right. Thus

$$24 : 8 : 6 = 12 : 4 : 3;$$

although $\quad 24 \div 8 \div 6 = \frac{1}{2}\quad$ and $\quad 12 \div 4 \div 3 = 1.$

Some authors define : as the precise equivalent of $\div$; but even they generally make some distinction in the use of the two signs.

V. Illustrations.

81. $-5 \quad -4 \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6$

Suppose that from the point marked 0 on this line I measure off equal distances to the right and left. These distances I call unit distances or steps, and the point 3 is three steps to the right of the origin of measurements, while the point -4 is four steps to the left of that origin.

Then the statements

$$-10 + 7 + 4 - 8 = 7 - 8 + 4 + 10 = 0 + 11 - 18 = -7$$

may be translated: "If, starting from a point 10 steps to the left of the origin, I go 7 steps right, then 4 right, then 8 left, I get to the same place as if, starting from 7 steps to the right of the origin, I go 8 steps left, then 4 right, then 10 left, or just the same as if, starting from the origin, I go 11 steps right and

then 18 left; in fact, by each of these routes I find myself 7 steps to the left of the origin."

Again, suppose I pay out \$10, take in \$7, take in \$4 more, and then pay out \$8. It will be the same, upon the whole, as if I had merely paid out \$7.

Still again, let A, B, C, D stand for four events. If I know that A happened 10 years ago, B 7 years after A, that C will happen 4 years after B, and finally that D happened 8 years before the time when C will happen; then D happened 7 years ago.

Let the student give other illustrations.

Imagine that after going 2 steps right and then 7 left, I find myself 10 steps to the left of the origin. Where was I?

I retrace my steps, going from 10 left, 7 steps right, and then 2 left, and I find myself 5 steps left from the origin. Or again, I notice that I have come upon the whole 5 steps left from the starting point, and retracing these, I get as before a point 5 steps left from the origin.

In symbols:

$$-10-(+2-7)=-10-2+7=-5,$$

and $$-10-(+2-7)=-10-(-5)=-10+5=-5$$

an illustration of association and sign-reversal.

Of the latter, here are others: less of westing is more of easting; less of spent is more of saved; to lighten one's burdens is to add to one's strength; taking away cold is making warm.

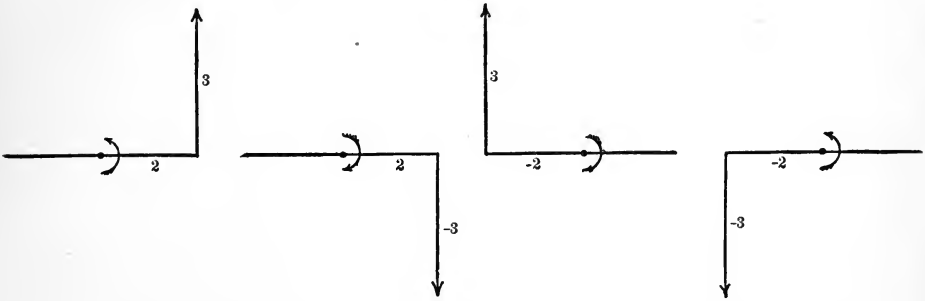
But there are problems where negative numbers are nonsense. A man cannot live a negative number of years. A pond cannot be -4 feet deep. A table cannot have -3 legs. No one is -6 feet tall.

82. If I twice repeat three steps to the right from the origin, I go 6 steps to the right. If I twice repeat 3 steps to the left, I go 6 steps to the left. If I twice retrace 3 steps to the right, I go 6 steps left. If I twice retrace 3 steps to the left, I go 6 steps right.

In symbols:

$$3 \times 2 = 6;\ (-3) \times 2 = -6;\ 3 \times -2 = -6;\ (-3) \times -2 = 6.$$

Again, I have a lever. Distances to the right on the lever are positive; to the left, negative. A pull up on the lever is positive; a pull down, negative. The unit pull is that of one pound one inch from the fulcrum, and is positive if it tends to lift the right-hand end of the lever.



Plainly, a pull up of 3 lbs., 2 in. to the right of the fulcrum is a 6-unit positive pull; a pull down of 3 lbs., 2 in. to the left of the fulcrum, is a 6-unit negative pull; a pull up 3 lbs., 2 in. to the left of the fulcrum, is also a 6-unit negative pull; and finally, a pull down of 3 lbs., 2 in. to the left of the fulcrum, is a 6-unit positive pull.

83. Suppose I am between a point 2 steps to the right of the origin and one 3 steps to the right of the origin. I can indicate my position by saying what part of the way I am from 2 to 3; and of course I can indicate my position by a fraction. Thus, the point $\frac{5}{2}$ is half way from 2 to 3. Such conventions serve to introduce fractions into all the illustrations that we have given. Notice, however, that just as negative numbers are sometimes non-sense, so are fractions. A polygon cannot have a fractional number of sides. A ball cannot be thrown $2\frac{1}{4}$ times. A surface has 2 dimensions, a solid 3; there is nothing between.

84. Not only are there distances to be expressed by fractions, but also distances which must be expressed by incommensurables. For example, the diagonal of a square whose

side was the unit distance would be $\sqrt{2}$. In fact, if we draw two lines at random we cannot take for granted that they have any common measure. Plainly, it is infinitely improbable that one should exactly measure the other: nor is it a whit more probable that a half, a third, a quarter, that any named fraction of the one should exactly measure the other.

If on the line of § 82 a point moves from 2 to 3, it passes through positions at all distances from the origin both fractional and incommensurable between 2 and 3. The same is true of distances from the fulcrum on our lever. Likewise, if in the latter problem the weight of pull is continuously increased, as it would be, very nearly, if due to water gradually flowing into a containing vessel; then, the weight of pull takes both fractional and incommensurable values. Thus the number expressing the pull in terms of the unit pull might be the product of two incommensurables. But as the pull varies directly with its weight and its distance from the fulcrum, these products must lie between the products of commensurables and be hemmed in by them as closely as one pleases.

VI. Growth and Rate.

85. We have now introduced the main part of the notation, the fundamental conceptions, and the material of elementary algebra. There remain for discussion the expression-numbers arising from the attempt to take even roots of negatives.

Before entering upon that discussion, however, we shall exhibit some of the operations and their results in a new light.

86. Suppose x to be, in succession, all numbers from zero to *a-number-as-large-as-you-please*. We say x grows positively from zero (0) to *infinity* ($+\infty$). On the other hand, if x is, in succession, all numbers from 0 to $-\infty$, we say that it grows negatively.

Any positive number a is the result of a positive growth, say, is a positive growth; while $-a$ is the result of a negative growth. The sum of two growths is the result of the growth

from either that would have given the other from zero. We easily extend this to any number of growths. Thus,

$$OA + AB + BC + CD + DE + EF + FG + GH = OH.$$

Evidently this equation would still be true if we were to change the position of the letters upon the line in any imaginable way. Quite independent is it too of what we choose for a unit growth, and of whether any or all of the growths added are incommensurable with that unit growth.

Subtraction is included in addition and need not delay us.

87. Let y grow from zero so that always $y = ax$, where a is a positive number that does not grow. If x' and y' are two corresponding fixed values of x and y, then always

$$y - y' = a(x - x'), \quad \text{and} \quad a = \frac{y - y'}{x - x'}.$$

We say that *y grows with x at a uniform rate a.*

Did we have $y = ax + b$, where b is another constant, we should, as before, have $\dfrac{y - y'}{x - x'} = a$. The rate of growth of y compared to x is the same as before; but y grows from b, while x grows from zero. In other words, y has the start b of x, and keeps that start.

Did we have $y = -ax$ or $y = -ax \pm b$, we should say that y grew *against* x at the uniform rate a.

When $y = ax$, we say that y' is the result of y's growing from zero at the uniform rate a, while x grows from zero to x'.

88. Suppose now $y = x^2$, and that as before x' and y' are corresponding fixed values of x and y. As x grows from $-\infty$ to zero, y grows against x from $+\infty$ to zero; and as x grows on from zero to $+\infty$, y grows with x from zero to $+\infty$. What is the rate of growth?

Consider the fraction $\dfrac{y-y'}{x-x'}$. We have

$$\frac{y-y'}{x-x'}=\frac{x^2-x'^2}{x-x'}=x+x'.$$

Evidently it changes its value as x changes its value; is less than $2x'$ when x is less than x', and is greater than $2x'$ when x is greater than x'; while at the moment when the growth of x reaches x' and $\dfrac{y-y'}{x-x'}$ becomes $\dfrac{y'-y'}{x'-x'}$, its value is $2x'$. We call the expression $\dfrac{y'-y'}{x'-x'}$ the *rate-fraction* and say that it gives the rate of growth y compared to x when $x=x'$.

On the face of it, $\dfrac{y'-y'}{x'-x'}=\dfrac{0}{0}$, and might be anything you please: for $k\times 0=0$ and so $k=\dfrac{0}{0}$, no matter what number k may be. In our present problem $\dfrac{y'-y'}{x'-x'}$ is $2x'$ because of the law connecting the growths of x and y as $\dfrac{y-y'}{x-x'}$ becomes or grows to be $\dfrac{y'-y'}{x'-x'}$. In fact $\dfrac{y'-y'}{x'-x'}$ is hemmed in as close as you please by values of $\dfrac{y-y'}{x-x'}$.

Thus, if $x'=0$ and we imagine x to take the values ± 1, $\pm\dfrac{1}{10}$, $\pm\dfrac{1}{100}$, $\pm\dfrac{1}{1000}$, $\pm\dfrac{1}{1000000}$, $\pm\dfrac{1}{10^n}$, where n is as large as you please, is infinity, the corresponding values of $\dfrac{y-y'}{x-x'}$ are likewise ± 1, $\pm\dfrac{1}{10}$, $\pm\dfrac{1}{100}$, $\pm\dfrac{1}{1000}$, $\pm\dfrac{1}{1000000}$, $\pm\dfrac{1}{10^n}$; so that as x nears x' or zero from either side $\dfrac{y-y'}{x-x'}$ also nears zero. But at the same time it nears $\dfrac{y'-y'}{x'-x'}$, and so tells us

that $\frac{y'-y'}{x'-x'}$ is not "anything you please;" but, on the contrary, is the definite number zero.

Drop the accents from $\frac{y'-y'}{x'-x'}$ and it becomes $\frac{y-y}{x-x}$, a variable, the varying rate of growth of y compared to x; and does itself grow with x at the constant rate 2. At first when x is negative ∞ the rate of growth of y compared to x is against x at the rate $2 \times \infty$. (We mean by this that however large in absolute value the number may be by which x is expressed, that by which the rate of growth is expressed will be twice as large.) As x increases, becoming less negative, the rate likewise increases and twice as rapidly as x, becoming less and less against x; until, when x is zero, the rate is zero. At this instant, the rate changes from being against x to being with x, and from now on, increasing as before, twice as fast as x does, becomes $2 \times \infty$, when x becomes ∞.

At the beginning of this section we said that as x grew from $-\infty$ to zero, y grew from $+\infty$ to zero. It would have been more accurate to have said that y grew from $(-\infty)^2 = \infty^2$ to 0; that is, that y grew from a number as many times greater than the opposite of x as the opposite of x was times greater than unity, no matter how large the opposite of x might be.

89. Whatever the law may be by which the growth of y is connected with the growth of x, we define the rate of growth of y compared to x by this same fraction $\frac{y-y}{x-x}$, and the rate of growth when $x = x'$ by the fixed fraction $\frac{y'-y'}{x'-x'}$.

Let the student determine the rates for—

$y = 2x - 3$,	when $x' = -3, -2, -1, 0, 1, 2, 3$;
$y^2 = 2x$,	" $x' = -7, 0, +5$;
$y = 3x^2 - 5$,	" $x' = -11, -1000, +20$;
$y = x^3$,	" $x' = 2$;
$y^3 = 3x^4 + 2x^3 + x^2 + 1$,	" $x' = 0, 1$.

Also, in each of the above, what is the rate of x-growth compared to y-growth? In other words, what is the value of $\frac{x - x}{y - y}$?

If $y = 2x + 5$ and $z = y^2$, what is the rate of z-growth compared to x-growth, when $x' = 3$?

90. Suppose $a > 1$ and $y = a^x$; what is $\frac{y - y}{x - x}$?

When x involves an even root, is, say, $\frac{3}{2}$, y might be negative: these values of y we rule out. We are considering growths, and y does not *grow* from one value to another unless it takes in succession *all* values between the two. Though the values are as close together as you please and y takes hundreds of thou sands of millions of values between thm, it does not strictly grow from one to the other. Thus, above, y cannot *grow* from $-a^1$ to $-a^{\frac{3}{2}}$; for though it may have as many values as you please between these two, there are as many other intermediate values, for example $-a$, which y cannot have. On the other hand, y does grow from $a^{\frac{1}{2}}$ to $a^{\frac{3}{2}}$ when x grows from $\frac{1}{2}$ to $\frac{3}{2}$, for y then takes all values between $a^{\frac{1}{2}}$ and $a^{\frac{3}{2}}$.

Then, as x grows from $-\infty$ through 0 to $+\infty$, y grows from 0 through 1 to $+\infty$.

At $x' = 0$, $\frac{y' - y'}{x' - x'} = \frac{1 - 1}{0 - 0}$ and the inclusives are $\frac{a^h - 1}{h}$ and $\frac{a^{-h} - 1}{-h}$, where h is a positive number growing zero-ward, so that h and $-h$ are inclusives of zero.

To prove that $\frac{a^h - 1}{h}$ and $\frac{a^{-h} - 1}{-h}$ really are inclusives of $\frac{y' - y'}{x' - x'}$ for $x' = 0$; we notice, first, that they are what $\frac{y - y'}{x - x'}$ becomes when we put in turn $x = h$ and $x = -h$ while x' remains zero. We then show that there is no least value of $\frac{a^h - 1}{h}$, no greatest value of $\frac{a^{-h} - 1}{-h}$, and that always $\frac{a^h - 1}{h} > \frac{a^{-h} - 1}{-h}$.

There is no least value of $\frac{a^h - 1}{h}$ if, for $0 < k < h$,

$$\frac{a^k - 1}{k} < \frac{a^h - 1}{h}\text{; i.e., if } 0 < h - k - ha^k + ka^h.$$

When $k = 1$ and $h = 2$ the condition becomes $0 < 1 - 2a + a^2$, which is $0 < (1 - a)^2$, and true because all squares are positive. Again, when $k = 2$ and $h = 3$, the condition becomes $0 < 1 - 3a^2 + 2a^3$, or $0 < a(1 - a)^2 + (a^2 - 1)(a - 1)$; and this is true because $a(1 - a)^2$ and $(a^2 + 1)(a - 1)$ are separately greater than zero. In like manner we could prove the condition to be fulfilled when $k = 3$ and $h = 4$. It is better, however, to show that if true for k and h *any* two consecutive numbers, it remains true when we increase both k and h by unity:

that if $$0 < 1 - (k + 1)a^k + ka^{k+1},$$

then also $$0 < 1 - (k + 2)a^{k+1} + (k + 1)a^{k+2}.$$

This is easy enough, for the last expression is the sum of a times $1 - (k + 1)a^k + ka^{k+1}$, positive by the first inequality, and $(a^{k+1} - 1)(a - 1)$, the product of two positives.

Therefore the first inequality does involve the second, and the condition holding for $k = 1$ and $h = 2$ holds for $k = 2$, $h = 3$, for $k = 3$, $h = 4$, for $k = 4$, $h = 5$, . . ., for $k =$ any integer and $h = k + 1$. Still more does it hold if k and h are integral, and $h > k + 1$.

Suppose now k and h are fractions $\frac{p}{q}$ and $\frac{r}{q}$, where $r > p > 0$. The condition becomes $0 < r - p - ra^{\frac{p}{q}} + pa^{\frac{r}{q}}$, precisely what we had before with r, p, and $a^{\frac{1}{q}}$ in place of h, k, and a; and is true because these numbers fulfil all the conditions imposed upon h, k, and a.

$$\therefore \frac{a^k - 1}{k} < \frac{a^h - 1}{h} \text{ for } 0 < k < h$$

when k and h are commensurable. But, since the values of the expression when k and h are incommensurable are hemmed in by the values when k and h are commensurable, the inequality is always true, and there is no least value of $\frac{a^h - 1}{h}$.

Neither is there a greatest value of $\frac{a^{-h} - 1}{-h}$.

This will be found to hang upon the inequality

$$0 < h - k - ha^{-k} + ka^{-h} \text{ or } 0 < h - k - h\left(\frac{1}{a}\right)^k + k\left(\frac{1}{a}\right)^h \text{ for } 0 < k < h.$$

The demonstration will proceed precisely as before, except that where we had $(a^{k+1} - 1)(a - 1)$, we shall now get $\left(\frac{1}{a^{k+1}} - 1\right)\left(\frac{1}{a} - 1\right)$, the product of two negatives instead of the product of two positives.

Finally, $\frac{a^h - 1}{h} > \frac{a^{-k} - 1}{-k}$ for $h > 0$ and $k > 0$, whether $h > k$, $h = k$, or $h < k$.

$$\text{If } h = k, \quad \frac{a^{-k} - 1}{-k} = \frac{1}{a^h} \cdot \frac{a^h - 1}{h} < \frac{a^h - 1}{h}.$$

$$\text{If } h > k, \quad \frac{a^h - 1}{h} > \frac{a^k - 1}{k} > \frac{a^{-k} - 1}{-k}.$$

$$\text{If } h < k, \quad \frac{a^h - 1}{h} > \frac{a^{-h} - 1}{-h} > \frac{a^{-k} - 1}{-k}$$

Thus, as stated, $\frac{a^h - 1}{h}$ and $\frac{a^{-h} - 1}{-h}$ are inclusives of $\frac{y' - y'}{x' - x'}$ for $x' = 0$ and $y = a^x$.

91. To fix our ideas, suppose a is 2, so that $y = 2^x$; then $\frac{y' - y'}{x' - x'}$ for $x' = 0$ lies between $\frac{2^h - 1}{h}$ and $\frac{2^{-h} - 1}{-h}$; i.e., between $n(\sqrt[n]{2} - 1)$ and $n(\sqrt[n]{2} - 1) \div \sqrt[n]{2}$, where $n = \frac{1}{h}$. For

convenience in calculation, put $n = 1, 2, 4, 8, \ldots$ We shall find the rate fraction, call it r, hemmed in as follows:

$$
\begin{array}{lllll}
n = 1, & 1. & > r & > 0.5; \\
n = 2, & 0.83 & > r & > 0.58; \\
n = 4, & 0.76 & > r & > 0.64; \\
n = 8, & 0.73 & > r & > 0.66; \\
n = 16, & 0.71 & > r & > 0.67; \\
\ldots & \ldots & \ldots & \ldots \\
n = 4096, & 0.6931 & > r & > 0.6929;
\end{array}
$$

and the rate to the nearest thousandth is 0.693.

92. To find the rate when x' is different from zero we have $\dfrac{y' - y'}{x' - x'} = \dfrac{2^{x'} - 2^{x'}}{x' - x'}$. But when $h > 0$,

$$\frac{2^{x'+h} - 2^{x'}}{h} > \frac{2^{x'} - 2^{x'}}{x' - x'} > \frac{2^{x'-h} - 2^{x'}}{-h};$$

since this may be written,

$$2^{x'} \cdot \frac{2^h - 1}{h} > 2^{x'} \cdot \frac{1 - 1}{0 - 0} > 2^{x'} \cdot \frac{2^{-h} - 1}{2^{-h}}.$$

Therefore the ratio at $x' = 2^{x'}r = y' \times 0.693$. When $x' = -2, -1, 0, 1, 2$, this gives for the rate $\frac{0.693}{4}$, $\frac{0.693}{2}$, 0.693, 2×0.693, 4×0.693.

The varying rate of growth $\dfrac{y - y}{x - x}$ has always the ratio 0.693 to the varying y; i.e., grows with y at the uniform rate 0.693. We say that y grows *logarithmically* with regard to x, the $\log_2 y$; and the number 0.693, $= \dfrac{\text{rate of growth}}{\text{growing number}}$, we call

the *logarithmic* growth rate. Thus, 2^7 is the result of allowing y to grow from unity at the logarithmic rate 0.693 with regard to x growing from zero to 7. Or, dropping the y and x, we say that 2^7 is the result of unity's growth at the logarithmic rate 0.693 with regard to zero growing to 7.

93. But unity is also $(2^7)^0$ and $2^7 = (2^7)^1$. So 2^7 is likewise the result of unity's growth at another logarithmic rate with regard to zero growing to 1. What is this new logarithmic rate? Why, $\dfrac{(2^7)^h - 1}{h}$ when h becomes zero. We may otherwise write it $7 \cdot \dfrac{2^{7h} - 1}{7h}$; and as here the last factor is 0.693, the new rate is 7×0.693.

Equally well can we get 2^7 by unity's growth at the logarithmic rate 3×0.693 with regard to zero growing to $\frac{7}{3}$, or by unity's growth at the logarithmic rate $k \times 0.693$ with regard to zero growing to $\dfrac{7}{k}$.

Call $k \times 0.693$, r, and in place of 7 write b. We see at once that any number 2^b can be reached by unity's growth at any assigned logarithmic rate r with regard to zero growing to $\dfrac{b}{r} \times 0.693$.

Fixing the logarithmic rate r fixes the base by whose powering unity grows. The base is in fact $2^{r \div 0.693}$.

Of great importance is the base for unit logarithmic rate, $2^{1 \div 0.693} = 2^{1.46} \cdots$, which we call the *natural*, *hyperbolic*, or *Naperian* base, and denote by e. At once, because $\frac{3}{2} > 1.46 > \frac{4}{3}$, $2^{\frac{3}{2}} > e > 2^{\frac{4}{3}}$, or $2.8 > e > 2.5$. More accurately, e is the incommensurable 2.7182818. . . . It has been defined as *the base by whose powering unity grows at the logarithmic rate unity;* it is also *the result of letting unity grow at the logarithmic rate unity with regard to zero growing to one.* Compare § 87.

Prove that by the powering of any base a, unity grows at the logarithmic rate $\log_e a$; and also that a is the result of unity growing at the logarithmic rate unity with regard to zero growing to $\log_e a$.

The *natural* logarithms, that is, the logarithms to the base e of the numbers

1, 2, 3, 4, 5, 6, 7, 8, 9,

are

0.000, 0.693, 1.099, 1.386, 1.609, 1.792, 1.946, 2.079, 2.197.

Notice that $1.386 = 2 \times 0.693$ and $1.792 = 0.693 + 1.099$. Why these relations?

At what logarithmic rate does unity grow by the powering of 12, 15, 27, 2.5, $3\frac{1}{3}$, $\frac{1}{2}$?

What must be the growth of zero for the above bases that unity may grow to 20? to 10? to -5?

94. The conception at the end of § 97 may be used to approximate to the number e.

When x grows from 0 to $\frac{1}{n}$, e^x grows from 1 to $e^{\frac{1}{n}}$. Had e^x kept the rate of growth it had for $x = 0$, i.e. grown uniformly instead of logarithmically with regard to x, it would have grown to $1 + \frac{1}{n}$. As it has really grown ever faster and faster, $e^{\frac{1}{n}} > 1 + \frac{1}{n}$. On the other hand, $e^{-\frac{1}{n}} < 1 - \frac{1}{n}$.

$$\therefore \left(\frac{n}{n-1}\right)^n > e > \left(\frac{n+1}{n}\right)^n,$$

whatever positive number n may be. We may otherwise write the e limits

$$\left(\frac{n+1}{n}\right)^{n+1} > e > \left(\frac{n+1}{n}\right)^n.$$

Here the superior limit is just $\frac{1}{n}$th larger than the inferior, and we therefore get e to within less than an nth part of itself. For instance, if $n = 1000000$, we know that e does not differ, since e is less than 3, 3 units in the sixth decimal place from $1.000001^{1000000}$.

By exact parity of reasoning,

$$\left(\frac{n+x}{n}\right)^{n+x} > \left(\frac{n}{n-x}\right)^{n} > e^x > \left(\frac{n+x}{n}\right)^{n}.$$

E.g., $\quad 1.000003^{1000003} > e^3 > 1.000003^{1000000}.$

Because $e < 3$, we have

$$e^3 < 27, \quad 1.000003\ e^3 < 27.000081, \quad 1.000003^2\ e^3 < 27.000162,$$

and $\quad 1.000003^3\ e^3 < 27.000243.$

Consequently $1.000003^{1000003}$ exceeds $1.000003^{1000000}$ by less than 0.000243, and either limit comes still closer to e^3. By taking n large enough we can of course get e^3 to any desired degree of accuracy.

Now notice. Not only is e^x as nearly as you please $\left(1+\frac{x}{n}\right)^n$, when n is taken large enough, but also $e^x = \left(1+\frac{1}{n}\right)^{nx}$;

$$\therefore \text{ when } n \text{ is large } \left(1+\frac{1}{n}\right)^{nx} = \left(1+\frac{x}{n}\right)^{n}.$$

This result might have been foreseen, for

$$\left(1+\frac{1}{n}\right)^{nx} = \left(1+\frac{x}{nx}\right)^{nx} = \left(1+\frac{x}{n}\right)^{n};$$

since the x multiplying the two n's in the middle expression is meaningless, n being merely any sufficiently large number.

A continuation of this reasoning shows that, to any desired degree of approximation,

$$e^x = \left(1-\frac{1}{n}\right)^{-nx} = \left(1-\frac{x}{n}\right)^{-n},$$

and $\quad e^{-x} = \left(1-\frac{1}{n}\right)^{nx} = \left(1-\frac{x}{n}\right)^{n} = \left(1+\frac{1}{n}\right)^{-nx} = \left(1+\frac{x}{n}\right)^{-n}.$

In brief, $e^x = \left(1 + \frac{1}{n}\right)^{nx} = \left(1 + \frac{x}{n}\right)^n$ holds for x and for n either positive or negative, provided n is large in absolute value.

95. Rough examples of uniform and logarithmic growth are furnished by money put out at simple and at compound interest respectively. Did the interest come in not merely yearly or half-yearly, or even every day or minute or second, but all the time, the examples would be perfect. What we call the rate of interest corresponds, in the one case, to what we have called rate of uniform growth; in the other, to what we have called rate of logarithmic growth. In the one case, the money grows by equal amounts in equal times; in the other, it grows by equal multiples of itself in equal times, is equally multiplied in equal times.

The more often we compound, the yearly rate remaining unchanged, the greater will be the amount of a given sum of money put out for a fixed time. Show, however, that, no matter how often the compounding, \$100 at 10% per annum for 10 years could not amount to so much as \$271.83.

VII. Graphs.

96. We can picture to the eye some of the results of the preceding sections.

We represented positive and negative numbers by distances measured along a line to the right and left of a fixed point called the origin. Equally well are they represented by distances from the line upward and downward.

In the equation $y = \frac{1}{2}x$, represent x by a distance along the line, and y by a distance measured at once from the line and from the end of the distance x.

Thus below, if x is 4, y is the distance 2 of the point (4, 2) from 4, on the line of x's.

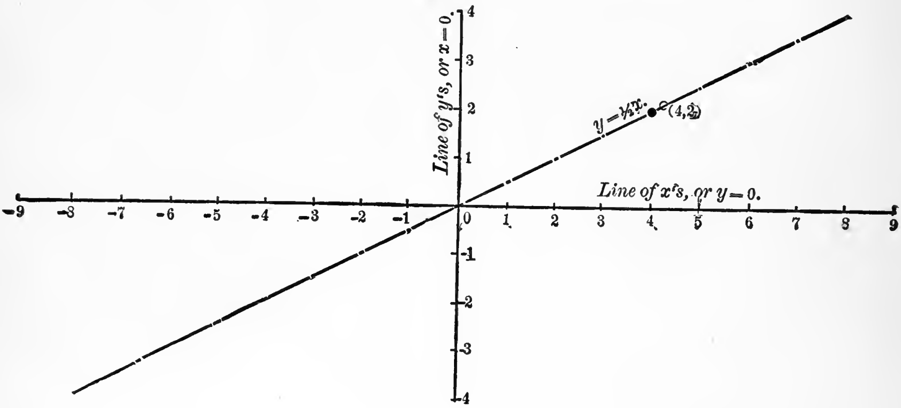
Equally well, of course, we can write the equation $x = 2y$, and say that if $y = 2$, x is the distance 4 of (4, 2) from the point 2 on the line of y's.

When x takes the series of values

$$-6,\ -5,\ -4,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,$$

y is $-3,\ -\frac{5}{2},\ -2,\ -\frac{3}{2},\ -1,\ -\frac{1}{2},\ 0,\ \frac{1}{2},\ 1,\ \frac{3}{2},\ 2,\ \frac{5}{2},\ 3,$

and is represented by the distances of the dots on the broken line from the line of x's.



By geometry, these dots are in one straight line, and the dots gotten by taking any value whatsoever of x would also lie on this line. Further, all points on the broken line are points (x, y), i.e. points such that the y of any one of them is half the x.

We call the line the *graph* of $y = \frac{1}{2}x$.

In $y = ax$, a is simply y for $x = 1$.

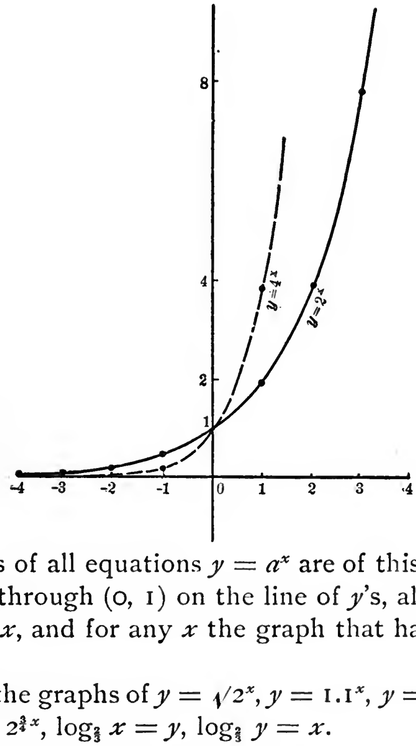
Let the student construct the graphs of $y = 5x$, $x = -2\frac{1}{2}y$, $y = 2x - 3$, $y = x\sqrt{2} + 1$, $2\frac{1}{3}x = -3\frac{1}{4}y$.

Having given a number of distances, show how by a graph to get a given multiple of all of them.

To construct the graph of $y = 2^x$, we have the points (x, y) as follows:

$$(-4, \tfrac{1}{16}), (-3, \tfrac{1}{8}), (-2, \tfrac{1}{4}), (-1, \tfrac{1}{2}), (0, 1), (1, 2), (2, 4), (3, 8), (4, 16).$$

The graph is sketched below. It is also the graph of $x = \log_2 y$.



The graphs of all equations $y = a^x$ are of this sort, if $a > 1$. They all pass through (0, 1) on the line of y's, all grow steeper for increasing x, and for any x the graph that has the greatest a is steepest.

Construct the graphs of $y = \sqrt{2}^x, y = 1.1^x, y = 1.01^x, y = 1^x$, $y = (\frac{1}{2})^x, y^{\frac{1}{2}} = 2^{\frac{3}{2}x}, \log_{\frac{1}{3}} x = y, \log_{\frac{1}{3}} y = x$.

97. In the graph of $y = ax$, $a_1 = \dfrac{y - y'}{x - x'}$ is called the *slope of the graph*, which is in this case a straight line. In the graphs of $y = x^2, y = a^x$, and in other curved graphs, $\dfrac{y' - y'}{x' - x'}$ is the *slope of the curves at* (x', y').

What is the slope of $y = 2x + 3$? the slope of $y = x^2$ at (0, 0) and where $x' = 2, 3, -1$? of $y = 2^x$ at (0, 1) and where $x' = 2, 5, -3$?

In general, the slope of a line joining arbitrary points (x, y), (x', y'), is $\dfrac{y - y'}{x - x'}$. If we put the points on a curved graph, the

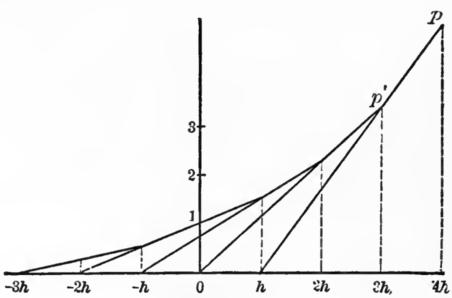
line cuts across the graph. If the points run together, ($y = y'$, $x = x'$), but remain on the graph, $\dfrac{y' - y'}{x' - x'}$ is the slope of a line touching the graph at (x', y').

Thus (compare § 93), 1, 0.83, 0.76, 0.73, 0.71 are respectively the slopes of lines cutting across the graph of $y = 2^x$ from the points (0, 1) to the points (1, 2), $(\frac{1}{2}, \sqrt{2})$, $(\frac{1}{4}, \sqrt[4]{2})$, $(\frac{1}{8}, \sqrt[8]{2})$, $(\frac{1}{16}, \sqrt[16]{2})$; while 0.693 is the slope of a line touching the graph at (0, 1). Our approximating to the value of the rate fraction is thus finding the slope of a cutting line as the points of cutting run together, making it a touching line.

98. Of course, when $y = a^x$ and $x - x' = h$, $\dfrac{y - y'}{x - x'}$ is $y' \cdot \dfrac{a^h - 1}{h}$. Suppose, as shown below, that we give x' the series of values

$$-2h, \quad -h, \quad 0, \quad h, \quad 2h, \quad 3h, \quad 4h, \quad 5h, \quad 6h, \quad 7h.$$

And further, let the line joining (x, y) with (x', y') cut the



line of x's where $x = k$. By geometry, $x' - k$, y', and the cutting line form a triangle similar to that formed by $x - k$, y, and the cutting line. They might be the triangles h, p', $3h$, and h, p, $4h$.

We have
$$\frac{x - k}{x' - k} = \frac{y}{y'} = a^h;$$

$$\therefore k = \frac{x'a^h - x}{a^h - 1} = x' - \frac{h}{a^h - 1},$$

and $$x' - k = \frac{h}{a^h - 1},$$

no matter what the value of x'.

In the figure we have supposed $\frac{h}{a^h - 1}$ to be $2h$, and starting with the point (0, 1) have found a number of points upon the graph.

Show that the line touching the graph of $y = 2^x$ at (0, 1) cuts the line of x's at (− 1.46, 0).

Prove that a line touching $x = \log_a y$ at (x', y') cuts the line of x's at $\left(x' - \frac{1}{\log_e a}, 0\right)$, i.e. at $(x' - \log_a e, 0)$.

The point (4, 3) is on $y = a^x$; what is the value of a?

In the figure just considered, suppose $h = 2$; then the graph of $y = \sqrt[4]{e^x}$ is touched at (0, 1) by the line joining $(-2h, 0)$ with (0, 1). Of course, all points of the graph, save (0, 1), lie wholly above that line; but the point $(1, \sqrt[4]{e})$ is only a little above, the point $(\frac{1}{2}, \sqrt[8]{e})$ still less above, and so on. Furthermore, were we to go out on the touching line till just under the point $(\frac{1}{2}, \sqrt[8]{e})$, and then start the construction, joining the point just reached with $(-3\frac{1}{2}, 0)$, the joining line would lie closer to $(1, \sqrt[4]{e})$ than our former line. By starting the construction closer and closer to 0, 1, we should get lines and points closer and closer to the graph of $y = \sqrt[4]{e}$, and as close as one pleased, when at a finite distance from the line of y's. This is a geometrical interpretation of $e^x = \left(1 + \frac{x}{n}\right)^n$.

Part Second.

DOUBLE NUMBERS.

I. Integral Double Numbers and the Simpler Operations.

99. We now consider the even roots of negatives referred to in § 87.

Assuming the law $a^m \times b^m = (a \times b)^m$, we have

$$\sqrt{-4} = \sqrt{4 \times -1} = \sqrt{-1 \times 4} = \sqrt{4} \cdot \sqrt{-1} = \sqrt{-1} \cdot \sqrt{4}$$
$$= \pm 2 \cdot \sqrt{-1} = \pm \sqrt{-1} \cdot 2.$$

Similarly,

$$\sqrt{-9} = \pm 3 \sqrt{-1},\ \sqrt{-16} = \pm 4 \sqrt{-1},\ \sqrt{-25} = \pm 5 \sqrt{-1}, \ldots,$$

and, putting i for $\sqrt{-1}$, we get the scheme

$$\ldots -5i,\ -4i,\ -3i,\ -2i,\ -i,\ 0,\ i,\ 2i,\ 3i,\ 4i,\ 5i \ldots$$

We call the new numbers *imaginaries*, *non-reals*, or *i-numbers*, while other numbers are *real* or *non-i*.

Of course we have, in the same way, i-numbers where the multipliers of i are fractional or incommensurable; but, for the present, we confine our attention to those with integral multipliers. We call them *integral i-numbers*, and say that their absolute value is the absolute value of the integral multipliers, thus making of i a new unit.

The distributive law for multiplication gives $ai + bi = (a + b)i$, which defines the addition of i-numbers.

For multiplication, we have

$$ai \times bi \times ci = \sqrt{-a^2} \times \sqrt{-b^2} \times \sqrt{-c^2} = \sqrt{-a^2 \cdot -b^2 \cdot -c^2}$$
$$= \sqrt{-a^2(-b^2 \cdot -c^2)} = \sqrt{-b^2 \cdot -c^2 \cdot -a^2} = \sqrt{-a^2 \cdot b^2 \cdot c^2 \cdot -1 \cdot -1}$$
$$= -1 \cdot \sqrt{-a^2b^2c^2}.$$

Consequently, $ai \times bi \times ci = ai \times (bi \times ci) = bi \times ci \times ai = -abci$, and the commutative and associative laws hold for the multiplication of i-numbers.

Further, *the absolute value of the product is the product of the absolute values of the factors; the product is an i or non-i number according as there are an odd or an even number of i-factors; and in determining its sign, each pair of i-factors gives a minus sign in addition to the minus signs before the several factors.*

Since $i \times i = -1$, $i \times -i = 1$, and $-i = \frac{1}{i}$. Therefore an i-number occurring as a divisor in any expression gives an opposite sign to that given by an i-number occurring as a multiplier.

The raising of i-numbers to integral non-i powers is included in the rules for multiplication and division. Nor is there any difficulty in taking integral non-i roots. As fractional powers are merely integral powers of integral roots, these are similarly disposed of.

Find single i or non-i numbers equivalent to the following expressions:

$$8i \,.\, 9i \div (i \,.\, 2i \,.\, 3i),\ (2i)^2 + (3i)^4 + 8 \div (i^6 - 4i)^4,$$

$$7i - (2i)^3 + (3i)^5 - i^{-5},\ \sqrt{i},\ \sqrt[3]{i},\ \sqrt[7]{i},\ \sqrt[6]{-64},\ \sqrt{-i},$$

$$i^{\frac{3}{2}},\ (-i)^{\frac{2}{3}},\ (-4)^{\frac{3}{2}},\ (-25)^{\frac{5}{2}},\ (-81)^{\frac{3}{4}},\ i^{-\frac{2}{3}},\ i^{-\frac{5}{4}},\ i^{-\frac{7}{5}},$$

$$ai \,.\, bi \div ci \,.\, di \div ei \,.\, fi.$$

100. To give a meaning to i-numbers, think of our definition of ordinary positive integers, names arbitrarily given to objects when counting a group of them.

Suppose we have several groups of objects. Number the groups as well as the objects in the groups. We then designate any object by two numbers: we say it is the object 2 in the group 3, the object 4 in the group 5, and so on. In other words, *to an object number we add a group number;* call this last an i-number, and we then say that the third object in the fourth group is the object $3+4i$.

All the objects 3 in all the groups themselves form a group; viz., the group

$$3+i, \quad 3+2i, \quad 3+3i, \quad 3+4i, \ \ldots$$

In that group $3+4i$ is the fourth object, and we are justified in writing $3+4i=4i+3$.

Of course, we can name both groups and objects by negative numbers as well as positive, and we can have starting or nul groups and objects.

The expression $a+bi$ we now call a *complex* or *double* number, of which a and b are the *non-i* and *i parts.*

To get to $3+4i$ we count forward from 0, the name of the nul object in the nul group, 3 object numbers, 4 group numbers. To get $(3+4i)+(5+7i)$ we count from either double number as we would count from zero to get the other.

$$\therefore (3+4i)+(5+7i)=5+7i)+(3+4i)=8+11i.$$

We see then that *two double numbers are equal if their i and non-i parts are separately equal, and the sum of any number of double numbers is that double number whose i and non-i parts are respectively the sums of the i and non-i parts of the double numbers added.*

Ordinary or non-i numbers are merely double numbers whose i parts are zero; while i-numbers are double numbers

whose non-i parts are zero. In symbols, $a = a + 0i$, and $ai = 0 + ai$.

Plainly, all the laws of addition, previously established, continue to hold.

101. Assuming multiplication distributive with regard to addition, and multiplying $1 + 2i$ by all double numbers, we get products that can be arranged as below.

$$
\begin{array}{ccccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\ldots, & 1-3i, & \underline{2-i}, & 3+i, & 4+3i, & 5+5i, & 6+7i, & 7+9i, & \ldots \\
\ldots, & \underline{-1-2i}, & \underline{0}\ , & \underline{1+2i}, & \underline{2+4i}, & \underline{3+6i}, & \underline{4+8i}, & 5+10i, & \ldots \\
\ldots, & -3-i, & \underline{-2+i}, & -1+3i, & 5i, & 1+7i, & 2+9i, & 3+11i, & \ldots \\
\ldots, & -5\ , & \underline{-4+2i}, & -3+4i, & -2+6i, & -1+8i, & 10i, & 1+12i, & \ldots \\
\ldots, & -7+i, & \underline{-6+3i}, & -5+5i, & -4+7i, & -3+9i, & -2+11i, & -1+13i, & \ldots \\
\ldots, & -9+2i, & \underline{-8+4i}, & -7+6i, & -6+8i, & -5+10i, & -4+12i, & -3+14i, & \ldots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}
$$

Here the product $(1 + 2i)(3 + 4i)$ is $2 + 11i$ in the 3d row down and 4th column right from 0.

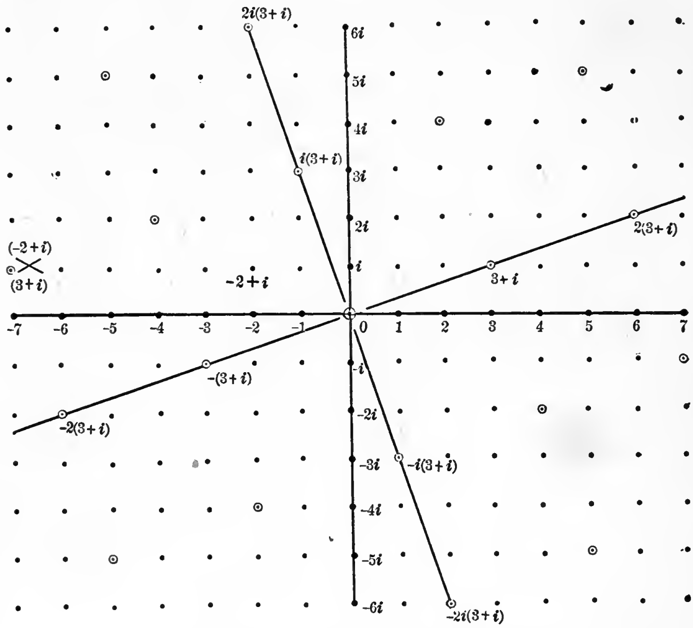
We have, in fact, a $1 + 2i$ *arrangement* or *system* in which the product is the 3d number in the 4th group. Thus, to multiply a number $a + bi$, by another $c + di$, we pick out the cth number in the dth group of the $a + bi$ system, or what is the same thing, the ath number in the bth group of the $c + di$ system.

It is as though we spoke of 6, the product of 2 and 3, as the second number in the third pair of numbers, or as the third number in the second triplet of numbers;

$$12,\ 34,\ 56, \quad \text{or} \quad 123,\ 456.$$

The significance of all this will become more apparent on adopting a simple geometric representation.

Suppose the objects to be dots, and the groups rows of them. Thus:



All the dots belong to the original or *standard* $1 + 0i$ system; those encircled, to the $3 + i$ system. We have indicated the dots $-2 + i$ and $(-2 + i)(3 + i)$.

The student may in like manner construct a diagram to show the $1 + i$ system, the $-1 + i$ system, the $2 + 0i$ or 2 system, the $0 + 3i$ or $3i$ system. Why do the dots of the first of these systems coincide with those of the second? Why do all dots belonging to the $3 - i$ and $2i - 7$ systems, belong also to the $-19 - i$ system? Why can no others be in that system?

The commutative law for the multiplication of double numbers is contained in the definition of a product. The student can show that the associative law also holds, and that multiplication is distributive with regard to addition when all the numbers involved are double.

Raising to integral non-i powers is done by repeated multiplication and does not require special consideration. Let the student construct diagrams showing the $2+i$, $(2+i)^2$, and $(2+i)^3$ systems; also, the i, i^2, i^3, and i^4 systems. In what do these last differ?

102. Of course, subtraction is a mere addition of opposites, while division is a guessing what to multiply one double number by to get another. It may or may not be integrably possible.

E.g., $(1+3i)\div(2+i)=1+i$; for $(1+i)(2+i)=1+3i$.

But there is no double number $x+iy$, with integers for x and y, such that $(1+i)(x+iy)=2+i$.

For expanding, we have

$$x-y+(x+y)i=2+i \text{ and } \therefore x-y=2 \text{ and } x+y=1;$$

equations which no integers satisfy.

Thus, $2+i$ is not integrably divisible by $1+i$.

Similar remarks apply to the extraction of integral non-i roots.

$$(1+5i)\div(3+2i)=?\ (1+7i)\div(2-i)=?\ (4+9)\div(2-3i)=?$$

$$25\div(3+4i)=?\ 17\div(i-4)=?\ 13\div(2i+3)=?\ \sqrt{3+4i}=?$$

$$\sqrt{13-12i}=?$$

Show that $\sqrt{(26-10i)^3}=(\sqrt{26-10i})^3$.

Prove the following not integrally possible:

$$(1+4i)\div(2-2i);\ (3-i)\div(2+3i);\ (7+i)\div(5-i).$$

II. Non-integral Double Numbers: Tensors and Sorts.

103. Fractions enter as with ordinary numbers. There, it will be remembered, they were numbers lying somehow between numbers already used. So double numbers with fractional parts lie somehow between the double integers. Better, the

fractional double numbers are such that their i and non-i parts lie between the i and non-i parts of integral double numbers. They are, if you please, names assigned to new objects interpolated into our groups and new groups interpolated into our systems.

Thus, in our diagram above, a dot placed between 1 and 2 on the initial line of the standard system would be marked by one-and-a-fraction; while a row of dots lying between the i and the $2i$ row would be a one-and-a-fraction row.

Manifestly, the interpolation can be carried to any extent, and the position of any point on the diagram marked as accurately as you please.

Let it then be carried out. Make the line of x's of § 96 the initial line, or line of non-i numbers; and the line of y's there, the line of i-numbers here. Finally, represent numbers with equal i parts by points equally distant, the same way, from the non-i line; and numbers whose non-i parts are equal by numbers equally distant, the same way, from the i-line.

In brief, put $$(x, y) = x + iy.$$

We have a representation of double numbers first used by a French mathematician, Argand, and frequently referred to as *the Argand diagram.*

Evidently, now as heretofore, *two double numbers are equal if their i and non-i parts are separately equal; and one double number lies between two other double numbers, if its i and non-i parts lie between the i and non-i parts of those others.*

104. Consider the number $\frac{a}{b} + \frac{c}{d}i$. This is

$$\frac{ad}{bd} + \frac{bc}{bd}i = \frac{1}{bd}(ad + bci).$$

Thus, all fractional double numbers are simple fractions of double integers.

In the $ad + bci$ system, $\frac{a}{b} + \frac{c}{d}i$ belongs to the initial

group, and the general type of all numbers belonging to that group is $\frac{e}{f}(ad + bci)$.

Two numbers are said to be of the *same sort* if they belong to the same group and if the multiplier $\frac{e}{f}$ has the same sign for both; but of *opposite sorts* if the number $\frac{e}{f}$ has opposite signs for the two. They are of *different sorts* if not belonging to the same group.

Show that $\frac{a}{b} + \frac{c}{d}i$ and $\frac{a}{c} + \frac{b}{d}i$ are of the same or opposite sorts according as b does or does not agree in sign with c.

Show that numbers of $\left\{\begin{matrix}\text{the same sort}\\ \text{opposite sorts}\end{matrix}\right\}$ have their points on the Argand diagram co-linear with the origin, and lying on $\left\{\begin{matrix}\text{the same side}\\ \text{opposite sides}\end{matrix}\right\}$ of the origin.

105. The product of $a + bi$ by $c + di$ is $ac - bd + (ad + bc)i$; and, obviously, the product of any number of the sort $a + bi$ by one of the sort $c + di$ will give one of the sort $ac - bd + (ad + bc)i$.

Suppose we were to divide $a + bi$ by $c + di$ and get the result $x + yi$. Then

$$a + bi = (x + yi)(c + di),$$

and

$$(a + bi)(c - di) = (x + yi)(c + di)(c - di) = (x + yi)(c^2 + d^2);$$

whence $x + yi = (a + bi)(c - di) \div (c^2 + d^2)$.

But also $x + yi = (a + bi) \div (c + di)$.

Now $c + di$ and $c - di$ are merely two double numbers that agree in everything save the signs of the i-parts. We call

such numbers *conjugates*, and we have just proved that *to multiply by either of two conjugates gives a number of the same sort as to divide by the other:* a remarkable and very useful theorem.

The number $+\sqrt{(\text{non-}i\text{ part})^2 + (i\text{-part})^2}$, by which if both of two conjugates are divided the results are reciprocals, is called the *modulus*, *absolute value*, or *tensor* of the numbers.

The student will see that when either the i-part is zero or the non-i part is zero, this agrees with previous definitions of absolute value.

Show that numbers whose tensors are equal have their points on the Argand diagram equidistant from the origin.

106. Let $a + ib$ be any number, and write $+\sqrt{a^2 + b^2} = m$. Further, let $\frac{a}{m} = p$ and $\frac{b}{m} = q$. Then

$$a + ib = m(p + iq), \quad \text{and} \quad p^2 + q^2 = 1.$$

The number is thus resolved into two factors: a quantity factor determining the absolute value, and a quality factor determining the sort. The latter, whose tensor is unity, we call a *complex unit*.

Let there be a second number $a' + ib' = m'(p' + iq')$, and consider the tensor of the sum, of the difference, of the product, and of the quotient of the two.

107. For the sum we have

$$a + a' + i(b + b'), \text{ with the tensor } \sqrt{(a + a')^2 + (b + b')^2}.$$

The tensor squared is

$$a^2 + b^2 + a'^2 + b'^2 + 2(aa' + bb') = m^2 + m'^2 + 2mm'(pp' + qq).$$

Were $pp' + qq' = 1$, this would plainly be the square of $m + m'$; and we should have the sum of the tensors for the tensor of the sum. This does happen when $p = p'$ and $q = q'$; that is,

when $a+ib$ and $a'+ib'$ are numbers of the same sort. Otherwise $pp'+qq'<1$, and *the tensor of the sum is less than the sum of the tensors.*

To see this, take account of the relations $p^2+q^2=p'^2+q'^2=1$, and write $pp'+qq'=k$. Evidently,

$$p^2-2pp'+p'^2+q^2-2qq'+q'^2-2-2k;$$

or
$$(p-p')^2+(q-q')^2=2(1-k).$$

The expression on the left is the sum of two squares, and positive, unless $p=p'$ and $q=q'$;

$$\therefore\ 1-k>0, \quad \text{and} \quad k=pp'+qq'<1.$$

On the other hand, $\qquad k>-1.$

For $\quad 0<(p+p')^2+(q+q')^2=2(1+k), \quad \text{and} \quad 1+k>0.$

$\therefore$ *the tensor of the sum is greater than the difference of the tensors* unless $p=-p'$ and $q=-q'$; that is, unless $a+ib$ and $a'+ib'$ are numbers of opposite sorts.

Show that the tensor of the $\left\{\begin{array}{c}\text{sum}\\ \text{difference}\end{array}\right\}$ of two numbers vanishes when and only when the numbers are $\left\{\begin{array}{c}\text{opposite}\\ \text{equal}\end{array}\right\}$.

Show that the tensor of the difference of two numbers exceeds the difference of their tensors, unless the numbers are of the same sort; while it is less than the sum of their tensors, unless the numbers are of opposite sorts.

Show that if two numbers are neither of the same nor of opposite sorts, their sum can be neither of the same sort as either of them nor of an opposite sort from either.

108. The tensor of a product is a simpler matter. We have

$$(a+ib)(a'+ib') = aa' - bb' + i(ab' + a'b)$$
$$= mm'\,[pp' - qq' + i(pq' + p'q)].$$

But $$(pp' - qq')^2 = p^2p'^2 - 2pp'qq' + q^2q'^2,$$

and $$(pq' + p'q)^2 = p^2q'^2 + 2pp'qq' + 2p'^2q^2;$$

and so

$$(pp'-qq')^2+(pq'+p'q)^2 = p^2(p'^2+q'^2)+q^2(p'^2+q'^2) = p^2+q^2 = 1.$$

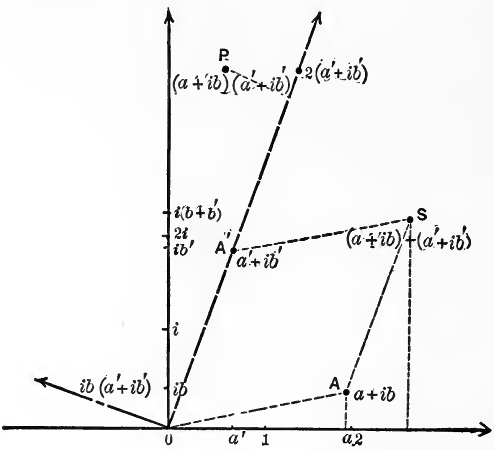
Thus $(pp' - qq') + i(pq' + p'q)$ is a complex unit, and $(a+ib)(a'+ib') = mm' \times$ that complex unit. In words: *the tensor of the product of two numbers is the product of their tensors.*

It goes without saying that *the ratio of the tensors of two numbers is the tensor of the ratio of the numbers.*

Let the student prove these.

The tensor of an integral power of a number is the integral power of the number's tensor.

If there be an integral root, its tensor is the integral root of the number's tensor.



Use the diagram representation. We go, from the nul-point, a right b up to $a+ib$ or A. Evidently $m, = +\sqrt{a^2+b^2}$,

is the distance from the nul-point to A. Likewise m' is the distance from the nul-point to A'; while $\sqrt{(a+a')^2+(b+b')^2}$ is the distance from the nul-point to $(a+ib)+(a'+ib')$ or S; o, A, A', and S are corners of a parallelogram; and of course $oA+AS > AS > oA - AS$.

Show that the distance AA' is $\sqrt{(a-a')^2+(b-b')^2}$.

To get the product point P, we go from the nul-point $(a'+ib')$-ward the distance am', and then $i(a'+ib')$-ward the distance bm'. Thus the distance of P from the nul-point is $\sqrt{a^2m'^2+b^2m'^2} = mm'$.

Prove that the triangles o, 1, A and $oA'p$ are similar.

Construct the sum and product points if $a+ib$ and $a'+ib$ are interchanged.

Construct the two difference points and the two ratio points.

Construct $(a+ib)^2$, $(a+ib)^3$, $(a+ib)^4$, $(a+ib)^{-1}$, $(a+ib)^{-2}$ (only one figure is necessary).

The problems are easily varied by changing the numbers a, a', b, and b'.

109. We have said that one double number lay between two others when the i and non-i parts of the one number lay between the i and non-i parts of the two others.

Instead of referring the numbers to the standard system, refer them to a $(p+iq)$ system $(p^2+q^2=1)$. In the place of non-i and i parts, we now have $p+qi$ and $ip-q$ parts.

Consistency requires that the definition of lying between shall be extended so as to include a reference to any possible system.

We say then that *one number lies between two others if p and q can be so chosen that the $p+iq$ and $ip-q$ parts of the one number shall lie between the $p+iq$ and $ip-q$ parts of the other numbers.*

Thus, consider the numbers

$$1+i, \quad 4+3i, \quad 6+2i.$$

Although 3 does not lie between 1 and 2, yet $4+3i$ does

lie between $1 + i$ and $6 + 2i$. For, refer the three numbers to a $\frac{3}{5} + i\frac{4}{5}$ system. They become respectively

$$\tfrac{7}{5}(\tfrac{3}{5}+i\tfrac{4}{5})-\tfrac{1}{5}(i\tfrac{3}{5}-\tfrac{4}{5}),\quad \tfrac{24}{5}(\tfrac{3}{5}+i\tfrac{4}{5})-\tfrac{7}{5}(i\tfrac{3}{5}-\tfrac{4}{5}),\quad \tfrac{26}{5}(\tfrac{3}{5}+i\tfrac{4}{5})-\tfrac{18}{5}(i\tfrac{3}{5}-\tfrac{4}{5})$$

and $\quad \frac{7}{5} < \frac{24}{5} < \frac{26}{5},\quad$ while also $\quad -\frac{1}{5} > -\frac{7}{5} > -\frac{18}{5}$.

To get clear ideas, let the student plot the above points on the Argand diagram, and then through the $1 + i$ and $6 + 2i$ points draw lines parallel to the initial and i lines of both the standard and the $\frac{3}{5} + i\frac{4}{5}$ system. He will thus get two rectangles having $1 + i$ and $6 + 2i$ for opposite corners. The rectangle that has a side parallel to the initial line of the standard system does not contain the point $4 + 3i$, while the other rectangle does.

To try every possible system to see whether one number lay between two others would be tedious. Take three numbers, $a_1 + ib_1$, $a + ib$, $a_2 + ib_2$. If a does lie between a_1 and a_2, and also b between b_1 and b_2, no test is necessary. Putting this in a slightly different form, if $(a_1 - a)(a - a_2) \geqq 0$, and also $(b_1 - b)(b - b_2) \geqq 0$, then $a - ib$ does lie between the other two numbers. Our problem then is to find a condition for lying between when either or both of the above products are negative.

Whatever the complex unit $p + iq$, we must have

$$a_1 + ib_1 = (pa_1 + qb_1)(p + iq) + (pb_1 - qa_1)(ip - q),$$

$$a + ib = (pa + qb)(p + iq) + (pb - qa)(ip - q),$$

$$a_2 + ib_2 = (pa_2 + qb_2)(p + iq) + (pb_2 - qa_2)(ip - q).$$

The multipliers of $p + iq$ and $ip - q$ now take the place of a_1, b_1, a, b, a_2, and b_2, above. It therefore follows that $a + ib$

does lie between $a_1 + ib_1$ and $a_2 + ib_2$ if, and only if, $p + iq$ can be so chosen that

$$[(a_1 - a)p + (b_1 - b)q][(a - a_2)p + (b - b_2)q] \geqq 0,$$

$$\text{say I} \times \text{II} \geqq 0,$$

and $$[(b_1 - b)p - (a_1 - a)q][(b - b_2)p - (a - a_2)q] \geqq 0,$$

$$\text{say III} \times \text{IV} \geqq 0.$$

Now, by the help of the relation $p^2 + q^2 = 1$, it will be found that

$$\text{I}^2 + \text{II}^2 + \text{III}^2 + \text{IV}^2 = (a_1 - a)^2 + (b_1 - b)^2 + (a - a_2)^2 + (b - b_2)^2 = A \text{ say};$$

and $$(\text{I} + \text{II})^2 + (\text{III} + \text{IV})^2 = (a_1 - a_2)^2 + (b_1 - b_2)_2 = B \text{ say}.$$

If $B < A$, $\text{I} \times \text{II} + \text{III} \times \text{IV} < 0$, and certainly either $\text{I} \times \text{II} < 0$, or else $\text{III} \times \text{IV} < 0$.

But $B < A$ unless $(a_1 - a)(a - a_2) + (b_1 - b)(b - b_2) \geqq 0$.

$\therefore a + ib$ does not lie between $a_1 + ib_1$ and $a_2 + ib_2$ except under the same condition.

Thus, $2 + 4i$ does not lie between $1 + i$ and $4 + 2i$; for $(-1)(-2) + (-3)2 = -4 < 0$.

To show that, if $(a_1 - a)(a - a_2) + (b_1 - b)(b - b_2 \geqq 0$, $p + iq$ can be so chosen that neither $\text{I} \times \text{II}$ nor $\text{III} \times \text{IV}$ shall be negative is easy. We merely chose it so that I, say, $= 0$. This makes $\text{I} \times \text{II} = 0$, and

because $$\text{I} \times \text{II} + \text{III} \times \text{IV} \geqq 0, \quad \text{III} \times \text{IV} \geqq 0.$$

The required value of $p + iq$ is $\dfrac{b_1 - b - i(a_1 - a)}{\sqrt{(b_1 - b)^2 + (a_1 - a)^2}}$.

Notice that B above is the tensor squared of $(a_1 + ib_1) - (a_2 + ib_2)$, while A is the sum of the squares of the tensors of $(a_1 + ib_1) - (a + ib)$ and $(a + ib) - (a_2 + ib_2)$. That B shall then be less than A means, on the Argand diagram, that $a + ib$ shall be somewhere within a circle constructed on the junction of $a_1 + ib_1$ and $a_2 + ib_2$ as a diameter.

110. Suppose $a_1 + ib_1$, $a + ib$, and $a_2 + ib_2$ to be such that I $\times$ II $\overline{\gtrsim}$ 0, no matter what the value of $p + iq$. Then $(a_1 - a)(a - a_2) \overline{\gtrsim} 0$, and also $(b_1 - b)(b - b_2) \overline{\gtrsim} 0$, as can be seen by putting first $p = 1$ and then $q = 1$ in the values of I and II. Furthermore, since $p + iq$ can become any complex unit, it may become $q - ip$. Now, when $p + iq$ changes to $q - ip$, I is changed to III and II to IV. Consequently, if always I $\times$ II $\overline{\gtrsim}$ 0, likewise always III $\times$ IV $\overline{\gtrsim}$ 0.

Since I and II are positive together and negative together, they must be zero together. For, let $p_0 + iq_0$ be a value of $p + iq$ that makes I vanish.

Then

$$\text{I} = (a_1 - a)p_0 + (b_1 - b)q_0 = 0, \text{ and II} = (a - a_2)p_0 + (b - b_2)q_0.$$

If II > 0, let $p + iq$ take a value such that II gets smaller, but still remains larger than zero (the student may show this possible).

Because the signs of the terms of I agree with the signs of the terms of II, I also gets smaller and therefore negative. I and II thus cease to agree in sign, and we could not have I $\times$ II > 0. Consequently II > 0 when I $= 0$; neither is II < 0.

We have then $\dfrac{a_1 - a}{b_1 - b} = \dfrac{-q_0}{p_0} = \dfrac{a - a_2}{b - b_2}$,

a condition to be satisfied by a, b, a_1, b_1, a_2 and b_2 in order that our supposition that I $\times$ II, and so III $\times$ IV, should never be negative, may be realized.

We say, in this case, that $a + ib$ lies *directly* between $a_1 + ib_1$ and $a_2 + ib_2$.

What condition besides $\dfrac{a_1 - a}{b_1 - b} = \dfrac{a - a_2}{b - b_2}$ must be fulfilled in order that $a + ib$ shall lie directly between $a_1 + ib_1$ and $a_2 + ib_2$? What that $a_1 + ib_1$ shall lie directly between $a + ib$ and $a_2 + ib_2$? that $a_2 + ib_2$ shall lie directly between $a + ib$ and $a_1 + ib_1$?

Show that one complex unit cannot lie directly between two other complex units.

Prove the following:

If one double number lies directly between two other double numbers, then a number directly between it and either of them lies directly between those two.

If a number lies between two others, but not directly so, a number between it and either of them need not lie between those two.

If a number lies between two others $\left\{\begin{matrix}\text{and}\\ \text{but not}\end{matrix}\right\}$ directly so, then a number between it and either of them, $\left\{\begin{matrix}\text{but not}\\ \text{and}\end{matrix}\right\}$ directly so, lies between the two numbers, but not directly so.

Work these out by the numerical conditions and then illustrate on the Argand diagram.

III. Complex Units and Non-*i* Powers.

111. In § 108 we proved that the tensor of the product of two numbers was the product of the tensors of the numbers. It followed that if there were an integral non-*i* root of a double number, its tensor was the integral root of the number's tensor. We now consider the possibility of this integral root. Evidently the possibility hangs upon whether a complex unit has such a root or not.

For simplicity, think of the square root and suppose

$$\sqrt{p+iq} = x + iy.$$

Then

$$(x+iy)^2 = p + iq, \quad \text{and so} \quad x^2 - y^2 = p.$$

But $\qquad T(p+iq) = 1. \qquad$ (T means "tensor of.")

$$\therefore\ T(x+iy) = 1 \quad \text{and} \quad x^2 + y^2 = 1.$$

Thus $x = \pm\sqrt{\frac{1+p}{2}}$, $y = \pm\sqrt{\frac{1-p}{2}}$, and for $p > 0$, $q > 0$, the required root is $\sqrt{\frac{1+p}{2}} + i\sqrt{\frac{1-p}{2}}$, or the opposite thereof, as may easily be verified.

By repetitions of this process we can, of course, get the fourth root, the eighth, any root whose index is an integral power of 2.

We have just seen that if both p and q are positive we can get a square root both of whose parts are positive. But no matter what the complex unit with which we start, a square root can be gotten whose i part is positive, and a fourth root both of whose parts are positive. Thus:

$$\sqrt{p+iq} = +\sqrt{\frac{1+p}{2}} + i\sqrt{\frac{1-p}{2}}, \text{ and so on;}$$

$$\sqrt{p-iq} = -\sqrt{\frac{1+p}{2}} + i\sqrt{\frac{1-p}{2}}, \text{ which is of the form } -p+iq;$$

$$\sqrt{-p+iq} = +\sqrt{\frac{1-p}{2}} + i\sqrt{\frac{1+p}{2}}, \text{ and so on;}$$

$$\sqrt{-p-iq} = -\sqrt{\frac{1-p}{2}} + i\sqrt{\frac{1+p}{2}}, \text{ which is of the form } -p+iq.$$

112. When p and q are both positive, $p^2 < p < \frac{1+p}{2}$, and

therefore $\quad p < \sqrt{\frac{1+p}{2}}$, while $q > \sqrt{\frac{1-p}{2}}$.

As we keep on extracting doubly positive square roots, the i part finally gets as near as you please to zero; while, at the same time, the non-i part gets as near as you please to unity.

Call such a resulting root $k + hi$, and consider the series of powers $k + hi$, $(k + hi)^2$, $(k + hi)^3$, . . . to $(k + hi)^n$, $= p + iq$.

They are all doubly positive complex units. Each has its i part greater and its non-i part less than that of the one before it. The tensor of the difference of two successive powers is constant and $= \sqrt{2(1 - k)}$.

To see the truth of this last statement, let $l + mi$ be one of the series of powers. The next one is $(k + hi)(l + mi)$, and their difference is $(l + mi)(k + hi - 1)$. Thus, at once, the tensor in question is

$$\begin{aligned} T(l + mi) \times T(k + hi - 1) &= 1 \times T(k + hi - 1) \\ &= \sqrt{(1 - k)^2 + h^2} = \sqrt{2(1 - k)}. \end{aligned}$$

It is plain that all doubly positive numbers $r + si$ with i part less than q must lie between two powers of $k + hi$; for the i and non-i parts of the number will lie between the i and non-i parts of two of the powers. (From our point of view, coinciding with either is merely an extreme case of lying between.) Since $(k + hi) = \sqrt[n]{p + qi}$, we can, by making n large, make h as near to zero, and so $\sqrt{2(1 - k)}$ as near to zero, as you please. It follows that $r + si$ is hemmed in as closely as you please by a power of a root of $p + qi$; i.e., $r + si$ can be expressed as closely as you please by a fractional power of $p + qi$; and, conversely, any power of $p + qi$ with index less than unity but greater than zero can be expressed as closely as you please by a number $r + si$.

113. We shall now show that all complex units whatsoever can be expressed as powers of $p + qi$, and that to every power of $p + qi$ corresponds a complex number.

In order to do this we establish three theorems:

1st. If $k > l > h$ and $k > m > h$, then the complex units $l + mi$, $-l + mi$, $-l - mi$, and $l - mi$, will on multiplication by $k + hi$ have both their i and non-i parts changed by more than $1 - k$, but by less than h.

Consider the first of the four. The changes in question are

$$l - kl + mh \quad \text{and} \quad km + hl - m.$$

Now $l - kl + mh > 1 - k$ if $l + k + mh > kl + k^2 + h^2$, which is obviously true since each term on the left is greater than the corresponding term on the right. Similarly,

$$km + hl - m > 1 - k, \quad \text{because} \quad km + hl + k > k^2 + h^2 + m;$$

$$l - kl + mh < h, \quad \text{because} \quad kl + mh < l + h;$$

and

$$km + hl - m < h, \quad \text{because} \quad km + hl < m + h.$$

Since no distinction is made in the conditions imposed upon l and m, these letters can be interchanged in all of the above inequalities without destroying their truth. But it will be found that the changes produced in $-l + mi$, $-l - mi$, and $l + mi$ on multiplication by $k + hi$ will either be the same as the above changes, or else derivable from them by the interchange of m and l. In fact, $l + mi$ itself, save for the interchange of m and l, is converted into the three other forms by merely passing to the i, i^2, and i^3 systems.

2d. If $1 \geqq g > k$ and $0 \leqq f < h$, then the complex units $f + gi$, $-g + fi$, $-f - gi$, $g - fi$ become, by the multiplication by $k + hi$, units $-f' + g'i$, $-g' - f'i$, $f' - g'i$, $g' + f'i$, where the numbers f' and g' fulfil the conditions $1 > g' \geqq k$, $0 < f' \leqq h$.

As before, to prove this for one of the given complex units is to prove it for all. We have

$$(f + gi)(k + hi) = fk - hg + i(fh + kg).$$

Now $fk - hg$ is largest when fk is largest, and smallest when hg is largest: is largest, therefore, when f is nearest h and g is nearest k, and smallest when $f = 0$ and $g = 1$. Hence

$$0 > fk - hg > -h, \quad \text{or} \quad 0 < hg - fk < h.$$

As for $fh + kg$, it is positive and largest when $fk - hg$ is least in absolute value, and least when $fk - hg$ is largest in absolute value.

$$\therefore 1 > fh + kg \geqq k.$$

Thus $hg - fk$ and $fh + kg$ can be replaced by numbers f' and g' conditioned as above.

3d. If the complex units $-f'+g'i$, $-g'-f'i$, $f'-gi$, $g'+f'i$, be multiplied by $k+hi$, we shall get respectively numbers $-l+mi$, $-l-mi$, $l-mi$, $l+mi$. We leave the proof for the student.

Start now with unity or $(k+hi)^0$, and form in succession the powers $k+hi$, $(k+hi)^2$, $(k+hi)^3$, . . . We shall get, in order, numbers $l+mi$, a number $f+gi$, a number $-f'+g'i$, numbers $-l+mi$, a number $-g+fi$, a number $-g'-f'i$, numbers $-l-mi$, a number $-f-gi$, a number $f'-g'i$, numbers $l-mi$, a number $g-fi$, a number $g'+f'i$, and finally numbers $l+mi$ again.

Now observe: if three different complex units agree in the sign of one of their parts, one of the three numbers must lie between the other two.

If they agree in the signs of both their parts, this is obvious enough. E.g., any number $l+mi$ lies between some two of those numbers $l+mi$ that we get by the powering of $k+hi$.

If the three agree in the sign of one of their parts only, the lying between may not be obvious. E.g., when $f>f'$, does a number $f+gi$ lie between a number $-f'+g'i$ and a number $l+mi$? Apply the test. We have for the quantity that is not to be less than zero

$$(-f'-f)(f-l)+(g'-g)(g-m).$$

If this expression can be diminished without becoming negative, it must be positive. But this is precisely what happens when for l we write f' and for $-m$ we write $+g'$. Each term of the expression is diminished, and their sum becomes $f'^2-f^2+g'^2-g^2$, which is zero. Similarly can be treated other sets of three complex units agreeing in the sign of only one of their parts.

Consequently all doubly positive complex units not lying between two doubly positive powers of $k+hi$ must lie between the last doubly positive power and the first negative-positive power, i.e. between the $f+gi$ and the $-f'+g'i$ number due to the powering. Similar statements apply to negative-positive, doubly-negative, and positive-negative units, and to the

numbers i, -1, $-i$, and 1 of which the non-expressed parts are ambiguous in sign. Thus the powers of $(k+hi)$ hem in all complex units and our thesis is proved. Of course, when we say "hemmed in," we take for granted that h shall be as small as you please, and so $\sqrt{2(1-k)}$, the tensor of the difference of the two successive powers between which any assigned complex unit must lie, as small as you please.

114. We have shown that all complex units whatsoever are hemmed in by powers of $k+hi$. Because $(k-hi)$ is the reciprocal of $k+hi$, and every complex unit is the reciprocal of some other complex unit, all complex units are likewise hemmed in by powers of $k-hi$. For the successive powers of $k-hi$, the i and non-i parts go through their changes in precisely the reverse order to that for powers of $k+hi$. In fact, we get, in order, positive-negative numbers, doubly-negative numbers, negative-positive numbers, and doubly-positive numbers.

Why can we not hem in complex units by successive powers of a negative-positive or a doubly-negative number?

115. Because all complex units are, as near as one pleases, powers of $k+hi$, any complex unit whatever is some power or other of any other complex unit. E.g.,

If $p+iq=(k+hi)^8$ and $r+si=(k+hi)^{11}$, $p+iq=(r+si)^{\frac{8}{11}}$.

Conversely, a complex unit can be found which comes as near as you please to any assigned power of any given complex unit $p+iq$.

Suppose we want $(p+qi)^{\frac{m}{n}}$. If n is a power of 2, no explanation is necessary. If not a power of 2, there are always two fractions, $\frac{a}{2^b}$, $\frac{a+1}{2^b}$, differing as little as you please, such that

$$\frac{a}{2^b}<\frac{m}{n}<\frac{a+1}{2^b},$$

and $(p+iq)^{\frac{m}{n}}$ is hemmed in by $(p+qi)^{\frac{a}{2^b}}$ and $(p+qi)^{\frac{a+1}{2^b}}$.

Calculate, for instance, $(\frac{3}{5}+i\frac{4}{5})^{\frac{11}{7}}$.

We have $\frac{12}{8} < \frac{11}{7} < \frac{13}{8}$,

and $\sqrt[8]{\frac{3}{5} + i\frac{4}{5}} = 0.9944 + 0.1057i.$

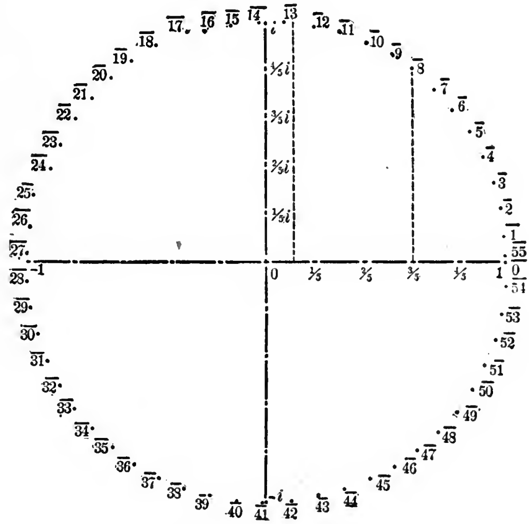
Whence $(\frac{3}{5} + i\frac{4}{5})^{\frac{12}{8}} = 0.179 \ + 0.984i,$

and $(\frac{3}{5} + i\frac{4}{5})^{\frac{13}{8}} = 0.074 \ + 0.997i.$

More closely, $\frac{1609}{1024} < \frac{11}{7} < \frac{1610}{1024}$; and we find that $(\frac{3}{5} + i\frac{4}{5})^{\frac{11}{7}}$ lies between $0.1126 + 0.9936i$ and $0.1135 + 0.9935i$.

Very accurately, $(\frac{3}{5} + i\frac{4}{5})^{\frac{11}{7}} = 0.1133739 + 0.9935537i.$

As an aid to clearness, we have plotted on the Argand diagram the points $(\frac{3}{5} + i\frac{4}{5})^0$, $(\frac{3}{5} + i\frac{4}{5})^{\frac{1}{8}}$, $(\frac{3}{5} + i\frac{4}{5})^{\frac{2}{8}}$, . . . to $(\frac{3}{5} + i\frac{4}{5})^{\frac{55}{8}}$, marking them $\overline{0}$, $\overline{1}$, $\overline{2}$, . . . to $\overline{55}$.



It will be noticed that 1 is at $\overline{0}$; i is between $\overline{13}$ and $\overline{14}$; -1, between $\overline{27}$ and $\overline{28}$; $-i$ between $\overline{40}$ and $\overline{41}$; and finally, 1 is also between $\overline{54}$ and $\overline{55}$.

Just to the right of $\overline{13}$ is the dot for $(\frac{3}{5} + i\frac{4}{5})^{\frac{11}{7}}$.

If we call $\frac{3}{5} + i\frac{4}{5}$, $k + hi$, then $\overline{0}$ to $\overline{12}$ are the $l + mi$ powers, $\overline{13}$ is the $f + gi$ power, $\overline{14}$ the $-f' + ig'$ power, $\overline{15}$ to $\overline{26}$ the $-l + mi$ powers, and so on.

The dots to represent the successive powers of $(\frac{3}{5}+i\frac{4}{5})^{\frac{1}{1024}}$ would be 128 times as near together as those for the powers of $(\frac{3}{5}+i\frac{4}{5})^{\frac{1}{8}}$, and could not be readily represented on our scale.

We have shown how any non-i power of a complex unit can be gotten with any desired degree of accuracy.

Any double number is a product *tensor* $\times$ *complex unit;* and therefore, *any power of a double number is the product power-of-tensor* $\times$ *power-of-complex-unit.*

Thus, a definite meaning is given to all non-i powers of double numbers. There remain for consideration the i and double-number powers. For this we prepare by extending to double numbers our ideas of growth.

IV. Growths, Rates, and Amounts.

116. A double number grows by the separate growths of its i and non-i parts. These growths may be connected by any law. In particular, they may be so connected that the i part grows uniformly with regard to the non-i part.

Thus, if $x+iy$, where $y=ax$, grows so that always y keeps equal to ax, then the rate of growth of y compared to x is a. If $x'+iy'$ is any number whatever reached by the growth, then the number $x-x'+i(y-y')$ is constantly of the sort $1+ia$. We say that the growth is a *uniform* one of the sort $1+ia$.

If, on the other hand, the rate of i growth compared to noni-i growth is not uniform, the number $x-x'+i(y-y')$ constantly changes its sort. We say that the $x+iy$ growth is of a varying sort, or, more briefly, is a *varying growth.*

When, for instance, $x+iy$ grows so that always $y=x^2$, the rate of i growth compared to non-i growth is $2x$, and itself grows with x at the rate 2. The sort of growth at $x'+iy'$ is given by $x-x'+i(y-y')$, and is therefore $1+i\dfrac{y-y'}{x-x'}$, which, for $y=y'$, becomes $1+2x'i$. Dropping accents, the growth at $x+iy$ is of the sort $1+2xi$, a number that changes with changing x.

On comparing the above with § 86 sequiter, it will be seen

that what we there called the graphs of $y = ax$, $y = x^2$, etc., are merely representations of the growths of $x + iy$ according to the laws $y = ax$, $y = x^2$, etc. Uniform and varying growths are represented respectively by straight lines and curves, while the sort of growth at a point $x + iy$ determines the law of growth of a tangent to the graph at the point in question.

117. Whatever be the law by which $x + iy$ grows, we can connect with it the growth of another number $u + iv$.

Suppose $y = ax$, and $u + iv = (c + id)(x + iy)$.

Then $u + iv$ as well as $x + iy$ has a growth of a uniform sort. In fact, the multiplication by $c + id$ merely changes the system, so that the growth of $u + iv$ would be represented by a straight line through zero and $(1 + ia)(c + id)$, just as the growth of $x + iy$ is represented by a line through zero and $1 + ia$.

The growth of $u + iv$ is thus of the sort $c - ad + i(d + ac)$, and the rate of i growth compared to non-i in $u + iv$ is $\dfrac{d + ac}{c - ad}$.

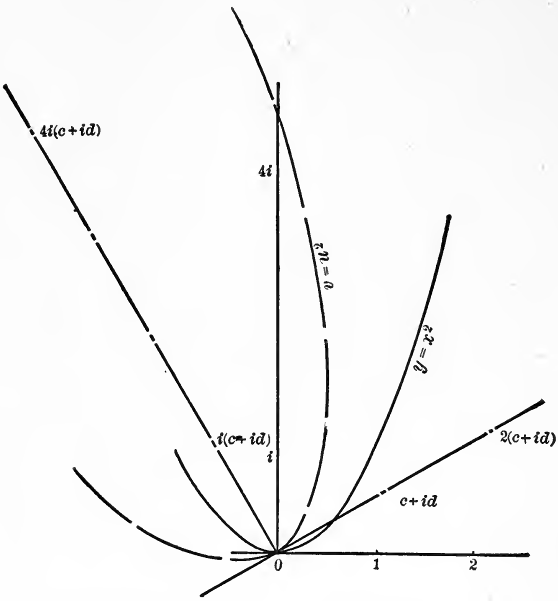
The rate of growth of $u + iv$ compared to $x + iy$ is of course $\dfrac{u + iv - (u' + iv')}{x + iy - (x' + iy')} = c + id$, a constant.

This last result is independent of how $x + iy$ grows. Does $x + iy$ grow so that always $y = x^2$? As before, the rate of $u + iv$ growth compared to $x + iy$ growth is $c + id$, and the graph of $u + iv$ in the $c + id$ system is the same, for this system, that the graph of $x + iy$ is for the standard system. These graphs are sketched in the annexed diagram. That of $x + iy$ with the standard reference-lines is drawn full; that of $u + iv$ with the transformed reference-lines broken. The arrows indicate the directions of positive growth in the two systems.

The rate of $ic - d$ growth compared to $c + id$ growth in the transformed graph is, of course, that of i compared to non-i growth in the original graph, viz. $2x$.

The rate of i with regard to non-i growth in the transformed graph is

$$\frac{u - u}{v - v} = \frac{cx^2 + dx - cx^2 - dx}{cx - dx^2 - cx + dx^2} = \frac{2cx + d}{c - 2dx}.$$



We leave it to the student to find the rate of growth of v compared to x, of u compared to y, of v compared to y, and of u compared to x.

118. From the two numbers $a+ib$ and $a'+ib'$, respectively, let there be uniform growths of the sorts $p+iq$ and $p'+iq'$. As usual, $p^2+q^2=p'^2+q'^2=1$.

Any number reached by the first growth can be written $a+mp+i(b+mq)$; while any number reached by the second is $a'+m'p'+i(b'+m'q')$.

If now $p+iq$ and $p'+iq'$ are not of the same or opposite sorts, we can always find one and only one pair of numbers m, m', satisfying the condition

$$a+mp+i(b+mq)=a'+m'p'+i(b'+m'q').$$

The numbers, in fact, are

$$m=\frac{p'(b-b')-q'(a-a')}{pq'-p'q} \quad \text{and} \quad m'=\frac{q(a-a')-p(b-b')}{pq'-p'q}.$$

In case $m = 0$, $p' + iq'$ is of the sort $a - a' + i(b - b')$, and $\frac{a - a'}{p} = \frac{b - b'}{q} = m'$; statements that remain true when accented and unaccented letters are interchanged.

When $p + iq$ and $p' + iq'$ are of the same or opposite sorts, either m and m' are non-finite ($m = \infty$, $m' = \infty$) and no number reached by the one growth can be reached by the other; or else you can take either what you please, provided the other is rightly paired with it.

The free choice is possible when either $p + iq$ and $p' + iq'$ are both of the same sort as $a - a' + i(b - b')$, both of an opposite sort from it, or one of them of the same sort and the other of the opposite sort. For this gives $m = \frac{0}{0}$, or $0 \times m = 0$; which is true no matter what number m may be.

Having taken m, we have for m' the value $\frac{a - a' + mp}{p'}$. In like manner, m' taken arbitrarily gives $m = \frac{a' - a + m'p'}{p}$.

The distinction between growths of the same and opposite sorts can of course be avoided by agreeing that a growth of an opposite sort is merely one of the same sort taken negatively.

119. The preceding investigation furnishes a simple test for one number's lying directly between two others.

That $p + iq$ and $p' + iq'$ shall each be of the sort $a - a' + i(b - b')$ is the same as saying that, if $x + iy$ be the number reached by the growths, then

$$\frac{b - y}{a - x} = \frac{y - b'}{x - a'}.$$

Now $x + iy$ is directly between $a + ib$ and $a' + ib'$ if, besides this, $b - y$ and $y - b'$ agree in sign.

But $$x + iy = a + ib + m(p + iq)$$
$$= a + ib + m\frac{a - a' + i(b - b')}{\sqrt{(a - a')^2 + (b - b')^2}};$$

which, if we put k for $m \div \sqrt{(a-a')^2 + (b-b')^2}$, becomes

$$(1+k)(a+ib) - k(a'+ib').$$

Thus $$y = (1+k)b - kb';$$

whence $b - y = k(b - b')$, and $y - b' = (1+k)(b - b')$.

That $b - y$ and $y - b'$ should agree in sign it is therefore necessary and sufficient that

$$k(1+k) < 0; \text{ i.e., } -1 < k < 0.$$

All the numbers, then, directly between $a + ib$ and $a' + ib'$ are contained in the form $l(a+ib) + l'(a'+ib')$, where l and l' are positive numbers whose sum is unity.

Furthermore, as the student can easily show, if, keeping $l + l' = 1$, we take l' negative, $a + ib$ lies directly between $a' + ib'$ and the number that we get; while, if we take l negative, $a' + ib'$ is the number between the other two.

From $7 + 8i$ and $2 - 3i$, respectively, are growths of the sorts $\frac{4}{5} - i\frac{3}{5}$ and $-\frac{12}{13} + i\frac{5}{13}$. What is the number reached by both?

From the same numbers, by growths of what sorts would $8 - 17i$ be reached?

Determine g in the following numbers so that they shall lie directly between $7 + 8i$ and $-13 + 42i$:

$$g + 21i, \quad -7 + gi, \quad 5 - gi, \quad 30i - g.$$

Represent these problems on the Argand diagram.

120. The number $a + ib$ is any number whatsoever. So also is $a + mp + i(b + mq)$. In order that it shall be, say, $c + id$, we require simply $a + mp = c$ and $b + mq = d$; whence

$$m = \frac{d-b}{q} = \frac{c-a}{p} = \sqrt{\frac{(d-b)^2 + (c-a)^2}{p^2 + q^2}} = \sqrt{(d-b)^2 + (c-a)^2}.$$

Thus m, p, and q are given in terms of a, b, c, and d, with the possibility of always taking m positive.

In other words, we can always have

$$m = T[c + id - (a + ib)].$$

We call it the tensor of uniform growth from $a + ib$ to $c + id$. The growth is evidently of the sort $c + id - (a + ib)$.

A uniform growth from $a + ib$ to a number $x + iy$, if not of the sort $c + id - (a + ib)$, must be followed by one also not of that sort to get from $x + iy$ to $c + id$. By § 107, the sum of the tensors of the two growths exceeds the tensor of a single uniform growth from $a + ib$ to $c + id$. We say that the single growth is more direct. *A fortiori*, the single growth is more direct than a chain of uniform growths from $a + ib$ to $c + id$ through numbers $x_1 + iy_1$, $x_2 + iy_2$, $x_3 + iy_3$, . . . $x_n + iy_n$, not lying directly between $a + ib$ and $c + id$.

121. Suppose that, in the chain of growths just suggested,

$$a < x_1 < x_2 < x_3 < \ldots < x_{n-2} < x_{n-1} < x_n < b;$$

and also

$$\frac{y_1 - b}{x_1 - a} > \frac{y_2 - y_1}{x_2 - x_1} > \frac{y_3 - y_2}{x_3 - x_2} > \ldots > \frac{y_{n-1} - y_{n-2}}{x_{n-1} - x_{n-2}} > \frac{y_n - y_{n-1}}{x_n - x_{n-1}} > \frac{d - y_n}{c - x_n}.$$

By addition and subtraction of numerators and denominators we get

$$\frac{y_1 - b}{x_1 - a} > \frac{y_2 - b}{x_2 - a} > \frac{y_3 - b}{x_3 - a} > \ldots > \frac{y_{n-1} - b}{x_{n-1} - a} > \frac{y_n - b}{x_n - a} > \frac{d - b}{c - a};$$

$$\frac{y_2 - y_1}{x_2 - x_1} > \frac{y_3 - y_1}{x_3 - x_1} > \frac{y_4 - y_1}{x_4 - x_1} > \ldots > \frac{y_{n-1} - y_1}{x_{n-1} - x_1} > \frac{y_n - y_1}{x_n - x_1};$$

together with similar chains of inequalities beginning

$$\frac{y_3 - y_2}{x_3 - x_2}, \quad \frac{y_4 - y_3}{x_4 - x_3}, \quad \frac{y_5 - y_4}{x_5 - x_4}, \quad \ldots \quad \frac{y_{n-1} - y_{n-2}}{x_{n-1} - x_{n-2}}, \quad \frac{y_n - y_{n-1}}{x_n - x_{n-1}}.$$

Also, we have

$$\frac{d-y_n}{c-x_n}<\frac{d-y_{n-}}{c-x_{n-1}}<\frac{d-y_{n-2}}{c-x_{n-2}}<\ldots<\frac{d-y_2}{c-x_2}<\frac{d-y_1}{c-x_1}<\frac{d-b}{c-a};$$

$$\frac{y_n-y_{n-1}}{x_n-x_{n-1}}<\frac{y_n-y_{n-2}}{x_n-x_{n-2}}<\frac{y_n-y_{n-3}}{x_n-x_{n-3}}<\ldots<\frac{y_n-y_2}{x_n-x_2}<\frac{y_n-y_1}{x_n-x_1}<\frac{y_n-b}{x_n-a};$$

together with similar chains of inequalities beginning

$$\frac{y_{n-1}-y_{n-2}}{x_{n-1}-x_{n-2}},\quad \frac{y_{n-2}-y_{n-3}}{x_{n-2}-x_{n-3}},\ \ldots\ \frac{y_3-y_2}{x_3-x_2},\quad \frac{y_2-y_1}{x_2-x_1}.$$

In brief, if we put for a, b, c, and d, respectively, x_0, y_0, x_{n+1}, and y_{n+1}, while letting l, k, j be three of the numbers $0, 1, 2, \ldots n+1$, with $l>j$, we have

$$\frac{y_k-y_l}{x_k-x_l}<\frac{y_k-y_j}{x_k-x_j}.$$

Any number directly between x_k+iy_k and $x_{k+1}+iy_{k+1}$ is $m(x_k+iy_k)+n(x_{k+1}+iy_{k+1})$, where m and n are positive and $m+n=1$. Therefore, unless $k+1=l$,

$$\frac{y_k-y_l}{x_k-x_l}>\frac{my_k+ny_{k+1}-y_l}{mx_k+nx_{k+1}-x_l}>\frac{y_{k+1}-y_l}{x_{k+1}-x_l};$$

since the middle ratio is $\dfrac{m(y_k-y_l)+n(y_{k+1}-y_l)}{m(x_k-x_l)+n(x_{k+1}-x_l)}$.

In like manner, unless $k=l+1$,

$$\frac{y_k-y_{l+1}}{x_k-x_{l+1}}>\frac{my_k+ny_{k+1}-y_{l+1}}{mx_k+nx_{k+1}-x_{l+1}}>\frac{y_{k+1}-y_{l+1}}{x_{k+1}-x_{l+1}}.$$

Then, if $r(x_l+iy_l)+s(x_{l+1}+iy_{l+1})$ is any number directly between x_l+iy_l and $x_{l+1}+iy_{l+1}$, it follows that

$$\frac{y_k-(ry_l+sy_{l+1})}{x_k-(rx_l+sy_{l+1})}<\frac{my_k+ny_{k+1}-(ry_l+sy_{l+1})}{mx_k+ny_{k+1}-(rx_l+sx_{l+1})}<$$

$$\frac{y_{k+1}-(ry_l+sy_{l+1})}{x_{k+1}-(ry_l+sy_{l+1})}.$$

Here notice that as m approaches unity and n approaches zero, the middle ratio approaches the left-hand ratio; while, when n approaches unity and m approaches zero, the middle ratio approaches the right-hand ratio. At the same time, $m(x_k + iy_k) + n(x_{k+1} + iy_{k+1})$ approaches, respectively, $x_k + iy_k$ and $x_{k+1} + iy_{k+1}$.

Now $r(x_l + iy_l) + s(x_{l+1} + iy_{l+1})$ is any number whatsoever on the chain of growths from $a + ib$ to $c + id$; and $m(x_k + iy_k) + n(x_{k+1} + iy_{k+1})$ is any other number on that chain.

Imagine, for the moment, a uniform growth joining these two numbers. The middle ratio above is that of the i to the non-i part of this growth. We are told then, by the inequalities, that as either of the two numbers changes its value along the chain of growths from $a + ib$ to $c + id$, this ratio likewise changes, growing ever smaller for a change of either number toward $c + id$, ever larger for a change toward $a + ib$.

Hence a uniform growth joining any two numbers on the chain of growths, but not itself forming part of the chain, cannot contain a third number on that chain.

122. Let there be a second chain of growths joining $a + ib$ to $c + id$, of the same character as the one just treated but through $z_1 + iv_1$, $z_2 + iv_2$, $z_3 + iv_3$, . . . , $z_m + iv_m$. Call the first the $x + iy$ chain; the second, the $z + iv$ chain.

Further, let every number on the $z + iv$ chain lie directly between $a + ib$ and some number on the $x + iy$ chain. In other words, let all the numbers on the $z + iv$ chain be of the type

$$r(a + ib) + s\left[m(x_k + iy_k) + n(x_{k+1} + iy_{k+1})\right].$$

Then, *the $z + iv$ chain is more direct than the $x + iy$ chain.*

To prove this, we need to show that just as the growth from $a + ib$ to $z_1 + iv_1$ will, if continued, contain some number on the $x + iy$ chain, so also will the growths from $z_1 + iv_1$ to $z_2 + iv_2$, from $z_2 + iv_2$ to $z_3 + iv_3$, from $z_3 + iv_3$ to $z_4 + iv_4$, and so on, every growth of the $z + iv$ chain containing, if continued, a number on the $x + iy$ chain.

Consider a series of uniform growths connecting $a + ib$, $x_1 + iy_1$, $x_2 + iy_2$, . . . , all to $z_k + iv_k$.

Since $z_k + iv_k$ is on a uniform growth from $a + ib$ to some number on the $x + iy$ chain, and likewise $z_{k+1} + iv_{k+1}$,

$$\frac{y_1 - b}{x_1 - a} \overset{=}{>} \frac{v_k - b}{z_k - a} > \frac{v_{k+1} - v_k}{z_{k+1} - z_k}$$

If $\dfrac{y_1 - b}{x_1 - a} = \dfrac{v_k - b}{z_k - a}$, then $z_k + iv_k$ is itself a number on the $x + iy$ chain; and since $\dfrac{d - v_{k+1}}{c - z_{k+1}} < \dfrac{v_{k+1} - v_k}{z_{k+1} - z_k}$, it follows that the growth from $z_k + iv_k$ to $z_{k+1} + iv_{k+1}$ is part of some uniform growth or other joining two numbers of the $x + iy$ chain.

If $\dfrac{y_1 - b}{x_1 - a} > \dfrac{v_k - b}{z_k - a}$, we may have $x_1 \gtreqless z_k$. If $x_1 \geqq z_k$, consider $m(a + ib) + n(x_1 + iy_1)$, where of course $m > 0$, $n > 0$, and $m + n = 1$.

Evidently $ma + nx$, as n grows from 0 to 1, will grow from a through z_k to x_1; but always the ratio $\dfrac{mb + ny_1 - b}{ma + nx_1 - a}$ is $\dfrac{y_1 - b}{x_1 - a}$. We have then

$$\frac{v_k - (mb + ny_1)}{z_k - (ma + nx_1)} > \frac{mb + ny_1 - b}{ma + nx_1 - a} > \frac{v_k - b}{z_k - b}, \text{ if } ma + nx_1 > z_k;$$

$$\text{but } \frac{mb + ny_1 - b}{ma + nx_1 - a} > \frac{v_k - b}{z_k - a} > \frac{v_k - (mb + ny_1)}{z_k - (ma + nx_1)}, \text{ if } ma + nx_1 < z_k;$$

$$\text{while, in either case, } \frac{v_k - (mb + ny_1)}{z_k - (ma + nx_1)} \overset{=}{>} \frac{v_k - y_1}{z_k - x_1}.$$

Thus $\dfrac{v_k - (mb + ny_1)}{z_k - (ma + nx_1)}$ can take all values between $\dfrac{v_k - b}{z_k - a}$ and $\dfrac{v_k - y_1}{z_k - x_1}$. Among these values is $\dfrac{v_{k+1} - v_k}{z_{k+1} - z_k}$.

Consequently, if $x_1 \geqq z_k$, the uniform growth connecting some number or other on the growth from $a + ib$ to $x_1 + iy_1$

with $z_k + iv_k$ is, together with the growth from $z_k + iv_k$ to $z_{k+1} + iv_{k+1}$, part of a uniform growth joining two numbers of the $x + iy$ chain.

But suppose $x_1 < z_k$, then $\dfrac{v_k - y_1}{z_k - x_1} \overline{<} \dfrac{v_{k+1} - v_k}{z_{k+1} - z_k}$. Repeating the previous reasoning, with a, b, x_1, y_1, replaced respectively by x_1, y_1, x_2, y_2, we find that, if $x_2 \overline{>} z_k$, the growth from $z_k + iv_k$ to $z_{k+1} + iv_{k+1}$ will yet be part of a uniform growth joining two numbers on the $x + iy$ chain. If $x_2 < z_k$, but $x_3 \overline{>} z_k$, the above statement still holds. Likewise does it if we can at last find, from among the numbers $x_1, x_2, x_3, \ldots, x_n$, c, a number larger than z_k. But certainly c is larger than z_k. Therefore, always, the growth from $z_k + iv_k$ to $z_{k+1} + iv_{k+1}$ is part of a uniform growth joining two numbers on the $x + iy$ chain.

Since the $z + iv$ chain is not identical with the $x + iy$ chain, at least one number determining the $z + iv$ chain is different from any number determining the $x + iy$ chain. Let, then, $z_k + iv_k$ be a number not identical with any of the numbers $a + ib$, $x_1 + iy_1$, $x_2 + iy_2$, . . . , $x_n + iy_n$, $c + id$. Because $\dfrac{v_k - v_{k-1}}{z_k - z_{k-1}} > \dfrac{v_{k+1} - v_k}{z_{k+1} - z_k}$, the three numbers $z_{k-1} + iv_{k-1}$, $z_k + iv_k$, $z_{k+1} + iv_{k-1}$, cannot possibly all be on one and the same growth of the $x + iy$ chain. Therefore at least one of the two growths of the $z + iv$ chain connecting these numbers is no part of a growth of the $x + iy$ chain. Continue this growth, then, till it meets the $x + iy$ chain, thus joining two numbers on that chain by a single uniform growth. A new chain is formed, made up of these three parts: the $x + iy$ chain from $a + ib$ to the first meeting number, the uniform growth from here to the second meeting number, the $x + iy$ chain from there to $c + id$. The new chain is more direct than the $x + iy$ chain. Call it a first shortening of that chain. Evidently this shortening enjoys all the properties stated as common to the $x + iy$ chain and the $z + iv$ chain. Furthermore, if not identical with the $z + iv$ chain, it is met by con-

tinuations of the growths of that chain; otherwise the $x+iy$ chain could not be met by these continued growths. Take, then, any number, z_l+iv_l, not determining the first shortening, and continue a growth of the $z+iv$ chain from it to meet that shortening. We get a second shortening. If this be not identical with the $z+iv$ chain, we can in like manner get a third shortening, and so on. But at the $(m+1)$st shortening, if not before, we reproduce the $z+iv$ chain. Thus the $z+iv$ chain, the result of successive shortenings of the $x+iy$ chain, is more direct than that chain.

To clearly fix in his mind the reasoning of this and the preceding section, the student should draw figures to represent the $x+iy$ and $z+iv$ chains, together with the typical joining lines to which reference is made. He will find that what is so difficult from the standpoint of pure algebra is geometrically a mere truism.

123. Let a growth of varying sort connect $a+ib$ with $c+id$, and let $z'+iv'$, $z''+iv''$, $z'''+iv'''$, be any numbers on that growth. Furthermore, let the growth be such that if only $z'<z''<z'''$, we shall have

$$\frac{v''-v'}{z''-z'}>\frac{v'''-v''}{z'''-z''}.$$

If then z_1+iv_1, z_2+iv_2, z_3+iv_3, . . . , z_m+iv_m, be numbers on the growth, and $a<z_1<z_2<z_3<\ldots<z_{m-1}<z_m<c$, the chain of growths gotten by directly joining these numbers in order will be of precisely the same nature as the $z+iv$ chain of the preceding section. We use the same name for it. The varying growth we, in like manner, call the $z+iv$ growth, and use $z+iv$ to denote a variable number on that growth. The growth is, if you please, the result of $(z+iv)$'s growing in a specified way from $a+ib$ to $c+id$.

The numbers $a+ib$, z_1+iv_1, z_2+iv_2, . . . , z_m+iv_m, $c+id$, are the only numbers common to growth and chain. For that the number $m(x_k+iy_k)+n(x_{k+1}+iy_{k+1})$, on the chain, should be also on the growth would contradict the inequalities used to define the character of the growth.

Suppose $z+iv$ to grow from $a+ib$ to x_k+iy_k, and then on to $c+id$. At the same time, the ratio $\frac{v-v_k}{z-z_k}$ grows from $\frac{b-v_k}{a-z_k}$ to $\frac{v_k-v_k}{z_k-z_k}$, and then on to $\frac{d-v_k}{c-z_k}$. The, by itself, meaningless expression $\frac{v_k-v_k}{z_k-z_k}$ is hemmed in by ratios that differ as little as we please, and so becomes a symbol of the limit of these ratios.

By § 118, there is a number, call it x_k+iy_k, such that

$$\frac{y_k-v_k}{x_k-z_k}=\frac{v_k-v_k}{z_k-z_k} \quad \text{and} \quad \frac{y_k-v_{k-1}}{x_k-z_{k-1}}=\frac{v_{k-1}-v_{k-1}}{z_{k-1}-z_{k-1}}.$$

Thus are defined a series of numbers x_1+iy_1, x_2+iy_2, x_3+iy_3, . . . , $x_{m+1}+iy_{m+1}$, forming, with $a+ib$ and $c+id$, the basis of an $x+iy$ chain. The only numbers common to it and the $z+iv$ growth are the numbers, $a+ib$, x_1+iy_1, x_2+iy_2, . . . , x_m+iy_m, $c+id$, also on the $z+iv$ chain.

The present $x+iy$ and $z+iv$ chains are related as were those of the preceding section; i.e., any number on the $z+iv$ chain lies between $a+ib$ and some number on the $x+iy$ chain. We leave the proof to the student. The proof established, it follows that the $x+iy$ chain is less direct than the $z+iv$ chain.

124. Take on the $z+iv$ growth between each pair of numbers z_k+iv_k, $z_{k+1}+iv_{k+1}$, a number $z_k'+iv_k'$, and then through $a+ib$, $z_0'+iv_0'$, z_1+iv_1, $z_1'+iv_1'$, z_2+iv_2, . . . , $z_m'+iv_m'$, $c+id$, form a $z'+iv'$ chain. From this chain form an $x'+iy'$ chain, as the $x+iy$ chain was formed from the $z+iv$ chain. Evidently the $x+iy$ chain is less direct than the $x'+iy'$ chain, that less direct than the $z'+iv'$ chain, and the $z'+iv'$ chain less direct than the $z+iv$ chain.

In the same way we can go on forever getting $z''+iv''$, $z'''+iv'''$, . . . , $z^{(n)}+iv^{(n)}$ chains, each less direct than the preceding one; and at the same time, $x''+iy''$, $x'''+iy'''$, . . . , $x^{(n)}+iy^{(n)}$ chains, each more direct than the preceding

one. But always the $z^{(n)}+iv^{(n)}$ chain is more direct than the $x^{(n)}+iy^{(n)}$ chain.

In other words, the sum of the tensors of the growths in the $x^{(n)}+iy^{(n)}$ chain approaches an inferior limit that either equals or exceeds a superior limit approached by the sum of the tensors of the growths in the $z^{(n)}+iv^{(n)}$ chain.

We can prove the two limits equal.

For convenience, call any two successive numbers common to the $z+iv$ growth and the $z^{(n)}+iv^{(n)}$ chain, $a+ib$ and $a'+ib'$.

Suppose $a'>a$, then $\dfrac{b-b}{a-a}>\dfrac{b'-b}{a'-a}>\dfrac{b'-b'}{a'-a'}$. For the ratios $\dfrac{b-b}{a-a}$ and $\dfrac{b'-b'}{a'-a'}$ we can put $\dfrac{q}{p}$ and $\dfrac{q'}{p'}$, with $p^2+q^2=p'^2+q'^2=1$ and $p'>p>0$. Let $a'-a=h$, and $\dfrac{q}{p}-\dfrac{q'}{p'}=k$.

The number of the $x^{(n)}+iy^{(n)}$ chain determined by the numbers $a+ib$ and $a'+ib'$ of the $z^{(n)}+iv^{(n)}$ chain is simply that number which can be reached both by a growth of the sort $p+iq$ from $a+ib$, and a growth of the sort $p'+iq'$ from $a'+ib'$. Using the notation of § 118, the number is, in fact,

$$a+mp+i(b+mq)=a'+m'p'+i(b'+m'q').$$

By § 120, m and m' are numerically equal to the tensors of the growths from $a+ib$ and $a'+ib'$. Call these tensors t and t' respectively. Then

$$t=\frac{p'(b'-b)-q'(a'-a)}{p'q-q'p}=\frac{h}{p}\cdot\frac{\dfrac{b'-b}{a'-a}-\dfrac{q'}{p'}}{k},$$

and $$t'=\frac{q(a'-a)-p(b'-b)}{p'q-q'p}=\frac{h}{p'}\cdot\frac{\dfrac{q}{p}-\dfrac{b'-b}{a'-a}}{k}.$$

The uniform growth from $a + ib$ to $a' + ib'$ and forming part of the $z^{(n)} + iv^{(n)}$ chain has the tensor

$$+\sqrt{(a'-a)^2 + (b'-b)^2} = h \, . \sqrt{1 + \left(\frac{b'-b}{a'-a}\right)^2} = l, \text{ say.}$$

Because of the inequalities $\frac{q}{p} > \frac{b'-b}{a'-a} > \frac{q'}{p'}$, we have

$$\frac{h}{p} > l > \frac{h'}{p'}.$$

Thus l lies between numbers differing by less than

$$h\left(\frac{1}{p} - \frac{1}{p'}\right).$$

Now $$\frac{1}{p} - \frac{1}{p'} = \left(\frac{1}{p^2} - \frac{1}{p'^2}\right) \div \left(\frac{1}{p} + \frac{1}{p'}\right).$$

But

$$\frac{1}{p^2} - \frac{1}{p'^2} = \frac{p'^2 - p^2}{p^2 p'^2} = \frac{(p^2 + q^2)p'^2 - (p'^2 + q'^2)p^2}{p^2 p'^2} = \frac{q^2}{p^2} - \frac{q'^2}{p'^2},$$

and $$\frac{1}{p} + \frac{1}{p'} > \frac{q}{p} + \frac{q'}{p'}.$$

$$\therefore \frac{1}{p} - \frac{1}{p'} < \left(\frac{q^2}{p^2} - \frac{q'^2}{p'^2}\right) \div \left(\frac{q}{p} + \frac{q'}{p'}\right) = \frac{q}{p} - \frac{q'}{p'} = k,$$

and the l inclusives differ by less than kh.

Observe that the last factors in the values of t and t', above, are less than unity, while the sum of the numerators of these factors is k. It follows at once that

$$t = \frac{l}{k}\left(\frac{b'-b}{a'-a} - \frac{q'}{p'}\right) - \text{something less than } kh,$$

$$t' = \frac{l}{k}\left(\frac{q}{p} - \frac{b'-b}{a'-a}\right) + \text{something less than } kh,$$

and $t + t' = l +$ something less than kh.

As formerly, suppose $a + ib$ and $c + id$ to be the terminal numbers of the $z + iv$ growth. The $x^{(n)} + iy^{(n)}$ chain is made up of growths having the tensors $t_1, t_1', t_2, t_2', t_3, t_3'$, . . . The $z^{(n)} + iv^{(n)}$ chain is made up of growths having the tensors l_1, l_2, l_3, . . . By what we have just shown,

$$t_1 + t_1' - l < k_1h_1, \quad t_2 + t_2' - l_2 < k_2h_2, \quad t_3 + t_3' - l_3 < k_3h_3, \ldots$$

Therefore the two chains differ by less than

$$k_1h_1 + k_2h_2 + k_3h_3 + \ldots \leqq K(c - a),$$

where K is the largest of the numbers k_1, k_2, k_3, . . .

Now $c - a$ is a fixed number, and K can be made as small as you please. Consequently $K(c - a)$ is as small as you please, and the $x^{(n)} + iy^{(n)}$ limit is identical with the $z^{(n)} + iv^{(n)}$ limit. We call this limit the *amount* of the $z + iv$ growth.

It may be objected that by different distributions of the numbers on the varying growth we could get different final limits. This, however, is easily shown to be impossible. For if Z' and X' are the amounts of the less and more direct chains determined by any distribution, and Z'' and X'' the corresponding amounts for any other distribution, while Z and X are the amounts determined by taking for our $z + iy$ and $x + iy$ numbers all the numbers contained in either and each distribution, we shall have

$$Z' < Z < X < X', \quad \text{and also} \quad Z'' < Z < X < X''.$$

Consequently the limit approached by increasing the number of numbers in either distribution is identical with that approached by increasing the number of numbers in the combined distribution.

Of course, instead of showing that the difference of the two limits must be less than $K(c - a)$, we could have shown that it was less than $\left(\frac{d - d}{c - c} - \frac{b - b}{a - a}\right)H$, where H is the largest of the numbers h_1, h_2, h_3, . . .

125. No essential difference in treatment is necessary for a varying growth in which the ratio $\frac{y-y}{x-x}$ continually increases with increase of x. Observe also that, if the growth is a non-varying one, the method is still applicable, but superfluous; since, in that case, the amount of growth does not differ from the tensor of the growth.

All growths whatsoever can be broken up into parts for which either $\frac{y-y}{x-x}$ increases with increase of x, or decreases with increase of x, or remains constant. The sum of the amounts of growth of the parts is the total amount of growth.

This entire investigation could have been carried through referring all the numbers to a $p + iq$ instead of a standard system, and in place of $\frac{y-y}{x-x}$, the ratio of the vanishing differences of the i and non-i parts of consecutive numbers on the varying growth, substituting the ratio of the vanishing differences of the $ip - q$ and $p + iq$ parts of those numbers.

V. Logarithmic Growths and Double-number Powers.

126. We have seen (§ 113) how $(p + iq)^n$, by the growth of n from zero, could become any complex unit whatever: how, indeed, passing once and only once through each and every complex unit, it could grow to $(p + iq)^0 = 1$, the starting value. What is the total amount of growth?

Consider that portion of the growth in which both the i and non-i parts of the growing number are positive. That is to say, let $x + iy$, keeping $x^2 + y^2 = 1$, grow from $0 + i$ to $1 + 0i$. As we shall presently see, the growth is such that $\frac{y-y}{x-x}$ decreases with increasing x. For convenience in calculation, we so take the numbers $x_1 + iy_1$, $x_2 + iy_2$, . . . , that the tensor of the difference of each two adjacent numbers shall be the same. For example, if between i and 1 we take only the number $x_1 + iy_1$, this shall be $i^{\frac{1}{2}} = \sqrt{\tfrac{1}{2}} + i\sqrt{\tfrac{1}{2}}$: if we take three

numbers, these shall be $x_1 + iy_1 = i^{\frac{3}{4}}$, $x_2 + iy_2 = i^{\frac{1}{2}}$, $x_3 + iy_3 = i^{\frac{1}{4}}$. Of course we always take doubly-positive roots, so that these last numbers are

$$\tfrac{1}{2}(\sqrt{2-\sqrt{2}}+i\sqrt{2+\sqrt{2}}),\ \tfrac{1}{2}(\sqrt{2}+i\sqrt{2}),\ \tfrac{1}{2}(\sqrt{2+\sqrt{2}}+i\sqrt{2-\sqrt{2}}).$$

After this fashion, interpolate 1023 numbers. They will be

$$x_1 + iy_1 = i^{\frac{1023}{1024}},\ x_2 + iy_2 = i^{\frac{1022}{1024}},\ \ldots,\ x_k + iy_k = i^{\frac{1024-k}{1024}},\ \ldots,$$
$$x_{1023} + iy_{1023} = i^{\frac{1}{1024}}.$$

The student may show that $x_k + iy_k = y_{1024-k} + ix_{1024-k}$, so that $x_{1023} + iy_{1023} = y_1 + ix_1$.

We have

$$T[(x_{k+1} + iy_{k+1}) - (x_k + iy_k)] = T\left(i^{\frac{1024-k-1}{1024}} - i^{\frac{1024-k}{1024}}\right)$$
$$= T(1 - i^{\frac{1}{1024}} = T[1 - (y_1 + ix_1)] = +\sqrt{2(1 - y_1)}.$$

Thus, the amount of growth is $1024\sqrt{2(1 - y_1)}$.

Let the student show that $y_{512} = \frac{1}{2}\sqrt{2}$, $y_{256} = \frac{1}{2}\sqrt{2+\sqrt{2}}$, $y_{128} = \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}$, . . . , $y_k = \frac{1}{2}\sqrt{2 + 2y_{2k}}$, and finally that

$$y_1 = \tfrac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}}}$$

In actually performing the calculation we begin at the right and proceed toward the left. If a sufficient number of decimal places be used, it will be found that

$$y_1 = 0.999998823449, \qquad x_1 = 0.0015339802,$$
$$\sqrt{2(1 - y_1)} = 0.0015339806, \quad 1024\sqrt{2(1 - y_1)} = 1.5707961.$$

The number of which $1024\sqrt{2(1 - y_1)}$ is an approximate value is symbolized by $\frac{\pi}{2}$, so that π is very nearly 3.1415922.

To find the error in the value of π we calculate, by the method of § 124, the error in taking for the growth from i to

$x_1 + iy_1$ the tensor of that growth. This multiplied by 1024 is the total error in the growth from i to 1. We have

$$\frac{y - y_1}{x - x_1} = \frac{y^2 - y_1^2}{x^2 - x_1^2} \cdot \frac{x + x_1}{y + y_1} = -\frac{x + x_1}{y + y_1} \gtrless -\frac{x_1}{y_1}, \text{ if } x_1 \gtrless x > 0.$$

But $\dfrac{y - y_1}{x - x_1} = \dfrac{y_1 - y_1}{x_1 - x_1}$, if $x = x_1$; and consequently $\dfrac{y_1 - y_1}{x_1 - x_1}$ $= -\dfrac{x_1}{y_1}$.

Likewise, $\dfrac{y - y}{x - x} = -\dfrac{x}{y}$, $\dfrac{y_0 - y_0}{x_0 - x_0} = -\dfrac{x_0}{y_0} = -\dfrac{0}{1} = 0$, and $\dfrac{y_{1024} - y_{1024}}{x_{1024} - x_{1024}} = -\dfrac{1}{0} = -\infty$.

(Thus, as x grows from 0 to 1, $\dfrac{y - y}{x - x}$ decreases from 0 to $-\infty$.)

If for a, b, a', b', of § 124, we write x_0, y_0, x_1, y_1, the t and t' of § 124 become

$$t = \frac{x_1}{1} \cdot \frac{\dfrac{y_1 - 1}{x_1} + \dfrac{x_1}{y_1}}{\dfrac{x_1}{y_1}}, \quad t' = \frac{x_1}{y_1} \cdot \frac{0 - \dfrac{y_1 - 1}{x_1}}{\dfrac{x_1}{y_1}}.$$

$$\therefore t = t' = \frac{1 - y_1}{x_1} \quad \text{and} \quad t + t' = \frac{2(1 - y_1)}{x_1} = 0.0015339810.$$

The error is thus less than 0.0000000004 for the growth from i to $x_1 + iy_1$, and less than 0.0000005 for the growth from i to 1. Therefore the amount of growth certainly lies between 1.5707961 and 1.5707966. Consequently

$$3.1415922 < \pi < 3.1415932.$$

More accurate calculation gives $\pi = 3.14159265 \ldots$.

When unity grows through all complex units around to unity again, the total amount of growth is made up of the amounts of growth from 1 to i, from i to -1, from -1 to $-i$,

and from $-i$ to unity. Each of these growths has the same amount as that from i to 1. The total amount of growth is therefore 2π.

127. The expression i^n when n starts to grow from zero, does itself start to grow from unity. What is the rate of growth of i^n compared to n? When n grows from 0 to ± 1, i^n grows from 1 to $\pm i$; and, moreover, if during the growths n takes a value between n' and n'', i^n at the same time takes a value between $i^{n'}$ and $i^{n''}$. Consequently $\dfrac{i^0 - i^0}{0 - 0}$, the symbol for the rate of growth of i^n compared to n when $n = 0$, lies between $\dfrac{i^h - 1}{h}$ and $\dfrac{i^{-h} - 1}{-h}$ where $1 \overline{\overline{>}} h > 0$.

Put $$h = 1, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \ldots \tfrac{1}{1024}.$$

Then the rate $\dfrac{i^0 - i^0}{0 - 0}$ is approached, on the one hand, by

$$i - 1,\ 2(i^{\frac{1}{2}} - 1),\ 4(i^{\frac{1}{4}} - 1),\ 8(i^{\frac{1}{8}} - 1),\ \ldots,\ 1024(i^{\frac{1}{1024}} - 1);$$

on the other, by

$$1 - i^{-1},\ 2(1 - i^{-\frac{1}{2}}),\ 4(1 - i^{-\frac{1}{4}}),\ 8(1 - i^{-\frac{1}{8}}),\ \ldots,\ 1024\left(1 - i^{\frac{-1}{1024}}\right).$$

Now $1024\,(i^{\frac{1}{1024}} - 1) = 1024(y_1 + ix_1 - 1)$, where x_1 and y_1 have the meaning given them in § 126.

But

$$1024(y_1 + ix_1 - 1) = -1024\sqrt{\frac{1 - y_1}{2}} \cdot \sqrt{2(1 - y_1)} + 1024x_1 \cdot i$$

$$= \frac{\pi}{2}\sqrt{\frac{1 - y_1}{2}} + i\frac{\pi}{2},$$

since $1024\sqrt{2(1 - y_1)}$ and $1024x_1$ are each, very nearly, $\dfrac{\pi}{2}$.

In the same way

$$1024\left(1 - i^{\frac{-1}{1024}}\right) = \frac{\pi}{2}\sqrt{\frac{1 - y_1}{2}} + i\frac{\pi}{2}, \text{ very nearly.}$$

Were we to take h smaller and smaller, the expressions for which we have written $\frac{\pi}{2}$ would become more and more nearly $\frac{\pi}{2}$, while $\sqrt{\frac{1-y_1}{2}}$ would become more and more nearly zero, both approachings going on without limit. Thus, the non-i part of $\frac{i^0 - i^0}{0-0}$ cannot be ever so little either positive or negative, while the i part cannot differ from $\frac{\pi}{2}$. The growth rate, then, of i^n compared to n when $n = 0$ is $\frac{i\pi}{2}$.

In the same way $\frac{i^n - i^n}{n-n}$, the general rate of growth of i^n compared to n, is as near as one pleases $\frac{i^{n+h} - i^h}{n+h-n}$, when h approaches zero.

But $$\frac{i^{n+h} - i^h}{n+h-n} = i^n \cdot \frac{i^h - 1}{h} = i^n \cdot \frac{i\pi}{2}.$$

Thus the rate of growth divided by growing number is constant and equal to $\frac{i\pi}{2}$. Compare now §§ 92, 93.

The result just reached may be stated:

By the powering of i, unity or i^0 grows at the logarithmic rate $\frac{i\pi}{2}$.

But $i^1 = i$, and the amount of growth from unity is $\frac{\pi}{2}$. Likewise the amount of growth from unity to i^n is $\frac{n\pi}{2}$. Hence *for unity to grow at $\frac{i\pi}{2}$ logarithmic rate means to grow keeping tensor constant and the amount of growth from unity always $\frac{\pi}{2}$ times the power-index.*

Similarly, *for unity to grow at logarithmic rate i means to grow keeping tensor constant and the amount of growth from unity equal to the index of the power.*

Notice that as the growths start from unity their sorts are i, while in passing any complex unit $p + iq$ their sorts are $ip - q = i(p + iq)$: the sort of growth divided by growing number is always i. We say that the growths are of the *logarithmic sort i.*

The number by whose powering unity grows at the logarithmic rate i must be the number whose $\frac{\pi}{2}$th power is i; for i can be reached both by unity's growth at logarithmic rate $\frac{i\pi}{2}$ with regard to zero growing to 1, or by unity's growth at logarithmic rate i with regard to zero growing to $\frac{\pi}{2}$. Thus, the base for logarithmic rate i is $i^{\frac{2}{\pi}}$.

128. Furthermore, by analogy, *i can be reached by unity's growth at logarithmic rate unity with regard to zero growing i-ward to $\frac{i\pi}{2}$.*

This definition gives for the base by whose powering unity grows at logarithmic rate unity $i^{\frac{2}{i\pi}}$, since $\left(i^{\frac{2}{i\pi}}\right)^{\frac{i\pi}{2}} = i$. But otherwise this base is $e = 2.71828 \ldots.$

Consequently $\quad i^{\frac{-2i}{\pi}} = e, \quad e^{i} = i^{\frac{2}{\pi}}, \quad i^{i} = e^{-\frac{\pi}{2}};$

$$e^{ki} = i^{\frac{2k}{\pi}}, \quad i^{ki} = e^{-\frac{k\pi}{2}}, \quad (e^{k}i^{l})^{gi} = i^{\frac{2kg}{\pi}} \cdot e^{-lg\frac{\pi}{2}};$$

$$(e^{k}i^{l})^{f+gi} = e^{kf - lg\frac{\pi}{2}} \cdot i^{lf + 2\frac{kg}{\pi}}.$$

Now $e^{k}i^{l}$ is any double number whatsoever, and so is $f + gi$. We have therefore shown that *any double-number power of a double number is a double number.* E.g.,

$$(1 + i)^{1+i} = \left[\sqrt{2} \cdot \left(\frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}}\right)\right]^{1+i} = (e^{0.347} \cdot i^{\frac{1}{2}})^{1+i} = e^{-0.438} \cdot i^{0.730}.$$

Since $e^0 = 1$ and $e^{0.693} = 2$, $e^{-0.438}$, the tensor of the number, lies between $\frac{1}{2}$ and 1. Because $i^{0.730} = \left(i^{\frac{2}{\pi}}\right)^{1.147}$, the amount of growth from unity to get the number is 1.147. On the diagram of § 114 this result would be represented by a point rather more than $\frac{1}{2}$ from the nul-point on a line from the origin passing about midway between $\overline{9}$ and $\overline{10}$, 0.73 of the way from 1 to i on the growth through $\overline{1}, \overline{2}, \overline{3}, \ldots$

In like manner, calculate and plot $(1 - i)^{i-1}$, $(\frac{3}{5} + i\frac{4}{5})^{i\pi}$, $(4 - 3i)^{1-i}$, $(-1)^i$, 1^{-i}, $1^{i\pi}$, $(-i)^{-i}$.

129. In the last section we saw that unity growing at logarithmic rate i with regard to zero growing to 1, became $i^{\frac{2}{\pi}}$. Had it grown at the uniform rate i, it would have become $1 + i$. Similarly, had it grown uniformly at rate i with regard to zero growing to $\frac{1}{2}$, and then at the uniform rate $i(1 + \frac{1}{2}i)$ with regard to the further growth of $\frac{1}{2}$ to 1, it would have become $1 + \frac{1}{2}i + \frac{1}{2}i(1 + \frac{1}{2}i) = \left(1 + \frac{i}{2}\right)^2$. In like manner, if for the growths from 0 to $\frac{1}{n}$ to $\frac{2}{n}$ to $\frac{3}{n}$ to ... to $\frac{n-1}{n}$ to 1, unity grows at the successive uniform rates i, $i\left(1 + \frac{i}{n}\right)$, $i\left(1 + \frac{i}{n}\right)^2$, ..., $i\left(1 + \frac{i}{n}\right)^{n-1}$, it becomes $\left(1 + \frac{i}{n}\right)^n$.

If $n = 1\,000\,000$, the tensor of $1 + \frac{i}{n}$ is $\sqrt{1.000\,000\,000\,001}$, and the tensor of $\left(1 + \frac{i}{n}\right)^n$ is

$$1.000\,000\,000\,001^{500\,000} = \sqrt[2\,000\,000]{1.000\,000\,000\,001^{1\,000\,000\,000\,000}}.$$

Thus $T\left(1 + \frac{i}{n}\right)^n$ is very closely the 2 000 000th root of e. If we put $n = 1\,000\,000\,000\,000$, $T\left(1 + \frac{i}{n}\right)^n$ is still more closely the 2 000 000 000 000th root of e. Increasing n without limit will

increase without limit the index of the root we must take of e to get $T\left(1+\frac{i}{n}\right)^n$. Consequently, by making n large enough, $T\left(1+\frac{i}{n}\right)$, $T\left(1+\frac{i}{n}\right)^2$, $T\left(1+\frac{i}{n}\right)^3$, . . . , $T\left(1+\frac{i}{n}\right)^n$ are all unity.

The successive growths by which we have supposed unity to become $\left(1+\frac{i}{n}\right)^n$ have each for its tensor $\frac{1}{n}$th of the tensor of the number from which it is supposed to grow. Therefore the sum of the tensors of all these n growths is unity.

Since the chain of growths is through numbers on a varying growth from unity of logarithmic sort i, and since the tensors of the growths are as small as you please, the sum of the growths is as near as you please the amount of the varying growth from unity to $\left(1+\frac{i}{n}\right)^n$. But when a growth is of logarithmic sort i and the amount of growth from unity is unity, the number reached by the growth is $i^{\frac{\pi}{2}} = e^i$. Now e is $\left(1+\frac{1}{n}\right)^n$ and e^i is $\left(1+\frac{1}{n}\right)^{in}$. Therefore

$$\left(1+\frac{i}{n}\right)^n = \left(1+\frac{1}{n}\right)^{in}.$$

Also, of course,

$$\left(1-\frac{i}{n}\right)^n = \left(1+\frac{i}{n}\right)^{-n} = \left(1+\frac{1}{n}\right)^{-in} \text{ and } \left(1-\frac{i}{n}\right)^{-n} = \left(1+\frac{i}{n}\right)^n;$$

provided always that n is taken large enough.

130. Consider now the expression $\left(1+\frac{p+qi}{n}\right)^n$. Just as the numbers $\left(1+\frac{i}{n}\right)$, $\left(1+\frac{i}{n}\right)^2$, $\left(1+\frac{i}{n}\right)^3$, . . . , are on a growth of logarithmic sort i, so we should naturally look for the numbers $1+\frac{p+qi}{n}$, $\left(1+\frac{p+qi}{n}\right)^2$, $\left(1+\frac{p+qi}{n}\right)^3$, . . . , on a

growth of logarithmic sort $p+qi$, i.e. on a growth always $(p+qi)$-ward from the growing number. By assigning a proper value to k any number on such a growth is $e^{(p+qi)\frac{k}{n}}$. Our expectation will then be justified, if we can show that $\left(1+\frac{p+qi}{n}\right)^k = e^{(p+qi)\frac{k}{n}}$. This is easy. For

$$\left(1+\frac{p+qi}{n}\right)^k = \left(1+\frac{p}{n}\right)^k \left(1+\frac{qi \div \overline{1+p\div n}}{n}\right)$$

$$= e^{\frac{k}{n}p} \cdot e^{\frac{k}{n} \cdot \frac{qi}{1+p+n}} = e^{\frac{k}{n}(p+qi)};$$

since $\frac{qi}{1+p\div n}$ is nearer than anything to qi when n is large. Notice also that $T\left(1+\frac{p+qi}{n}\right)^k = e^{\frac{k}{n}p} = \left(1+\frac{p}{n}\right)^k$.

To get the amount of growth of $p+qi$ logarithmic sort from 1 to $e^{\frac{k}{n}(p+qi)}$, we add together the tensors of the successive growths from power to power of $1+\frac{p+qi}{n}$ till $\left(1+\frac{p+qi}{n}\right)^k$ is reached.

The 1st tensor is $T\frac{p+qi}{n} = \frac{1}{n}$;

" 2d " " $T\left(e^{\frac{p+qi}{n}} \cdot \frac{p+qi}{n}\right) = e^{\frac{p}{n}} \cdot \frac{1}{n}$;

" 3d " " $T\left[\left(e^{\frac{p+qi}{n}}\right)^2 \cdot \frac{p+qi}{n}\right] = e^{\frac{2p}{n}} \cdot \frac{1}{n}$;

. . . .

The kth tensor is $T\left[\left(e^{\frac{p+qi}{n}}\right)^{k-1} \cdot \frac{p+qi}{n}\right] = e^{\frac{k-1}{n}p} \cdot \frac{1}{n}$.

Consequently the sum is $\frac{1}{n}\left(1+e^{\frac{p}{n}}+e^{\frac{2p}{n}}+\ldots+e^{\frac{(k-1)p}{n}}\right)$.

At the same time that these growths are taking place the tensor of the growing number has the growths

$$\frac{p}{n},\ \frac{p}{n}e^{\frac{p}{n}},\ \frac{p}{n}e^{\frac{2p}{n}},\ \ldots,\ \frac{p}{n}e^{\frac{(k-1)p}{n}}.$$

But the first value of the tensor is 1 and the last $e^{\frac{kp}{n}}$, while its growth is of a non-i sort. Hence

$$e^{\frac{kp}{n}}-1=\frac{p}{n}\left(1+e^{\frac{p}{n}}+e^{\frac{2p}{n}}+\ldots+e^{\frac{(k-1)p}{n}}\right).$$

Comparing this sum of the growths of the tensor with the sum of the tensors of the growths above, we have for that sum

$$\frac{1}{p}\left(e^{\frac{kp}{n}}-1\right).$$

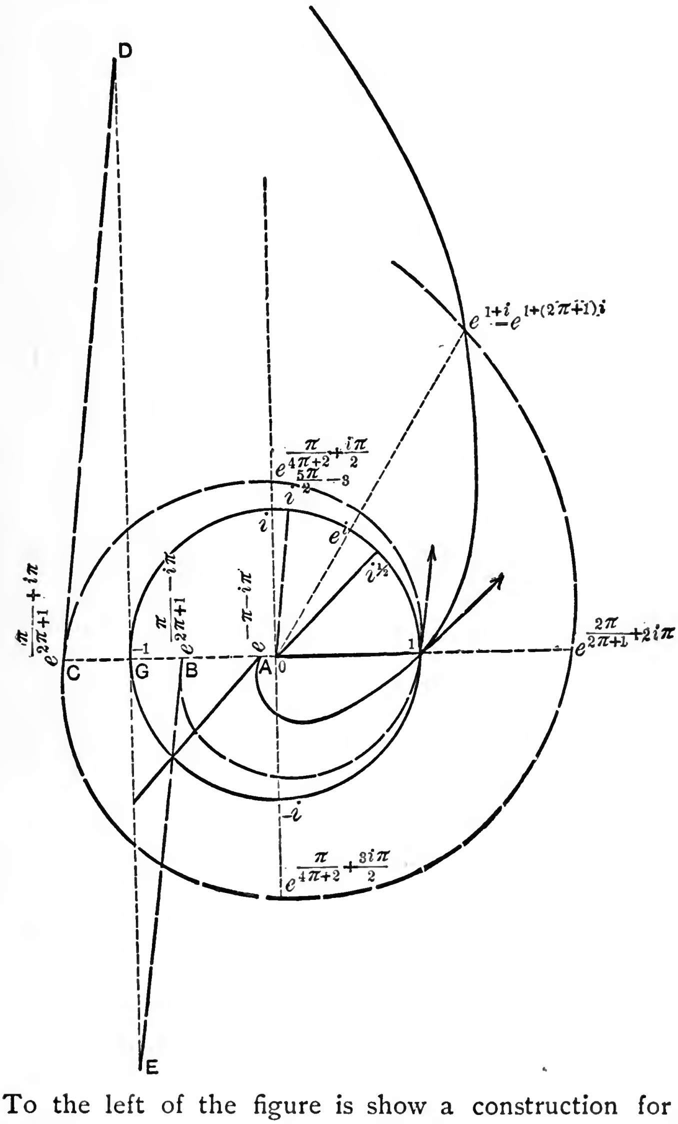
When $p=1$ and so the growth is of a non-i sort, this is $e^{\frac{k}{n}}-1$; while for $p=-1$, it is $1-\dfrac{1}{e^{\frac{k}{n}}}$: results which verify our formula. Again, if $p=0$, and the growth is of a pure i sort, the amount of growth is $\dfrac{\left(e^{\frac{k}{n}}\right)^0-1}{0}=\dfrac{k}{n}$, another verification.

Observe that whether the logarithmic rate be $p+qi$ or some non-i multiple of $p+qi$ in nowise affects the results. The amount of growth depends solely upon the logarithmic sort and the starting and terminal numbers of the growth.

Notice some geometric interpretations. The circle is the result of a growth of logarithmic sort i; the spiral drawn full, of a growth of logarithmic sort $1+i$; the spiral drawn broken, of a growth of logarithmic sort $1+(2\pi+1)i$. Several points on each growth are numbered. The student should satisfy himself that the numbering is correct.

The direction of the spiral growths at the start from unity is indicated by arrows. Notice that each spiral cuts all rays at the same angle that it there cuts the unit ray. The one

growth is always $(1 + i)$-ward, the other $[1 + (2\pi + 1)i]$-ward as to the ray from which it momentarily grows.



To the left of the figure is show a construction for the amount of growth. *BE* is the amount of growth on the broken

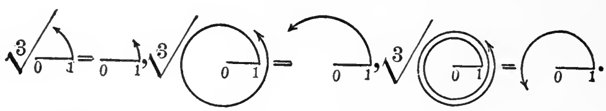
spiral from B to 1, DC the amount of growth from 1 to C, while AF is the amount of growth on the unbroken spiral from A to 1.

To prove the construction right take the last case. We have amount of growth $=\frac{1-e^{-\pi}}{1\div\sqrt{2}}=\frac{OG-OA}{AG\div AF}=AF.$

131. We have represented e^{1+i} as reached by two growths; one of the $1+i$, the other of the $1+(2\pi+1)i$ sort. Obviously, it is reached by all growths of the sorts $1+(2n\pi+1)i$ gotten by giving to n all integral values; for $e^{1+(2n\pi+1)i}=e^{1+i}(e^{2\pi i})^n$ $=e^{1+i}\cdot 1^n=e^{1+i}$. The $1+(2n\pi+1)$ spiral is such as to make n turns and $\frac{1}{2\pi}$th of a turn while the tensor of the growing number grows from 1 to e.

Suppose we wish the kth root of $e^{1+(2n\pi+1)i}$. By our ordinary rules, this should be $e^{\frac{1}{k}}\cdot e^{\frac{i}{k}}\cdot e^{\frac{2n}{k}i\pi}$. The first factor is the tensor of the root and independent of n. The last can be written $e^{2\frac{k'}{k}i\pi}$, where k' is the remainder after dividing n by k. Since k' can be any one of the numbers 0, 1, 2, 3, . . ., $k-1$, according to the value assigned to n, we thus get k and no more distinct values for the second factor. In other words, every root of $e^{1+(2n+1)i}$ can be reached by less than a single turn of some logarithmic growth from unity, and every kth root is on some one of the k growths of the logarithmic sorts $1+i,\ 1+(2\pi+1)i,\ 1+(4\pi+1)i,\ \ldots,\ 1+(2\cdot\overline{k-1}\cdot\pi+1)i.$

If the growth be specified the root is uniquely determined. Thus:



Now any double number is e^{a+bi}, where $b<2\pi$. The k distinct kth roots of this double number all have the tensor $e^{\frac{a}{k}}$, and lie on the k growths of the logarithmic sorts $a+bi$, $a+(2\pi+b)i$, $a+(4\pi+b)i'$, . . ., $a+(2\cdot\overline{k-1}\cdot\pi+b)i$.

Consequently each can be reached by less than a single turn of a logarithmic growth from unity.

Similar remarks apply to raising to fractional powers, the denominator of the fraction giving the number of roots. As we increase the denominator of the fraction, the number of roots forever increases and they approach closer and closer together. Remember now that we hem in incommensurables by fractions, only by indefinitely increasing the denominators of the fractions. It follows that an incommensurable power of a number, the growth on which it lies not being specified, is as near as one pleases, any number having the right tensor.

132. To say that any number $c + id$ can be reached from unity by an infinity of distinct logarithmic growths is the same as saying that every number has an infinity of logarithms to base e; for the logarithmic rate that determines the growth is nothing else than a logarithm of the number to the base e. In fact, if $a + ib$ is a logarithm of $c + id$, so also are $a + (2\pi + b)i$, $a + (4\pi + b)i$, . . ., $a + (2n\pi + b)i'$. That logarithm whose i part is less than 2π but not less than zero we call *the* logarithm of the number. On the growth determined by it lies that kth root of the number whose sort is between 1 and $e^{\frac{2i\pi}{k}}$. Unless otherwise specified, we take any fractional or incommensurable power of the number upon this same growth.

133. Usually, $\log_e(c + id)$ and $\log_e(c' + id')$ are numbers of different sorts, and so a logarithmic growth from unity containing one of them will not contain the other. We can, however, in general find three numbers k, p, q, $(p^2 + q^2 = 1)$ such that the growth from e^{ki} of a logarithmic sort $p + iq$ shall contain both $c + id$ and $c' + id'$. For, suppose $\log_e(c + id) = a + ib$ and $\log_e(c' + id') = a' + ib'$. Then $c' + id' = e^{a' - a + i\,b' - b} \cdot (c + id)$, and the wished-for growth is of the sort $a' - a + i\,(b' - b)$. This determines p and q. From

$$e^{ki} \cdot e^{g(p + iq)} = e^{a + ib}$$

we have $g = a \div p$, and thence $k = b - gq = b - a\dfrac{q}{p}$.

Calculate k, p, and q for $c + id = e^{1+i}$ and $c' + id' = e^{2+3i}$, drawing a diagram to show the results. Show that the method fails when and only when $c^2 + d^2 = c'^2 + d'^2$, and that then both numbers lie on a growth of logarithmic sort i from $\sqrt{c^2 + d^2}$.

134. Suppose we wish the $(c' + id')$th power of $c + id$, and that $c + id = e^{a+ib}$. The desired power is $e^{(a+ib)(c'+id')}$, and is the result accordingly either of unity's growing at the logarithmic rate $c' + id'$ with regard to zero growing $(a + ib)$-ward to $a + ib$, or of unity's growing at logarithmic rate $a + ib$ with regard to zero growing $(c' + id')$-ward to $c' + id'$. The growth makes the same angle at unity with the $\left\{ \begin{matrix} a + ib \\ c' + id' \end{matrix} \right\}$ logarithmic growth that the ray to $\left\{ \begin{matrix} c' + id' \\ a + ib \end{matrix} \right\}$ makes with the ray to unity.

Plot on the Argand diagram two numbers, and the power of each of them by the other.

135. Any number $e^{(a+ib)}$ is completely determined when we know the tensor e^a, $= r$ say, together with b giving the sort. For brevity, we write then $e^{a+ib} = r_b$: so that 1_b, when b grows from 0 to 2π, becomes in succession all complex units.

Just as $x + iy$ has its form of growth determined by a relation between x and y, so r_b has its form of growth determined by a relation between r and b.

If b is constant, then r_s is of a constant sort, and the growth is a uniform one; the same, in fact, as that of $x + iy$ for $x = ky$, where k is the ratio of the non-i to the i part of r_b. If, on the other hand, r is constant, the growth is the same as that of $x + iy$ when $x^2 + y^2 = r^2$.

Suppose $r = e^b$. Then the rate of growth of r compared to b is $\dfrac{e^{b} - e^b}{b - b} = e^b = r$. But always the b growth is i-ward to r growth; consequently the rate of growth of r_b compared to b is $r_b(1 + i)$, or the logarithmic rate of growth is $1 + i$. We thus get one of the logarithmic spirals of § 120.

Plot the growths of r_b for $r = b$, $r = b^2$, $r = \sqrt{b}$, $r = \dfrac{1}{b}$.

If $1_b = p + qi$, show the following equalities to be true:

$p - qi = 1_{-b}$, $\quad -p - qi = 1_{b+2\pi} = 1_{b-2\pi}$, $\quad p + qi = 1_{b+2n\pi}$
where n is integral,
$q - ip = 1_{b-\pi}$, $-q - ip = 1_{b-\pi}$, $i = 1_{\frac{\pi}{2}}$, $-1 = 1_{\pi}$, $-i = 1_{-\frac{\pi}{2}}$.

If also $1_{b'} = p' + iq'$, show that
$1_b \times 1_{b'} = 1_{b+b'}$, $1_b \div 1_{b'} = 1_{b-b'}$, $pp' - qq' + (pq' + p'q)\, i = 1_{b+b'}$.
Express in terms of p, q, p', q', the numbers 1_{2b}, 1_{3b}, $1_{\frac{b}{2}}$.

From § 126 we have

$$1_{\frac{\pi}{2048}} = 0.999998823449 + 0.00153398803i.$$

Calculate $1_{\frac{\pi}{1024}}$, $1_{\frac{3\pi}{2048}}$, $1_{\frac{\pi}{4096}}$, $1_{\frac{3\pi}{4096}}$.

In the calculation of § 126, in order to get $1_{\frac{\pi}{2048}}$, we had first to get the non-i parts of $1_{\frac{\pi}{4}}$, $1_{\frac{\pi}{8}}$, $1_{\frac{\pi}{16}}$, . . . Had we at the same time gotten the i-parts, we could then easily obtain, by the formulæ immediately above, $p + iq$ expressions of 1_b for all values of b, exact multiples of $\dfrac{\pi}{2048}$.

VI. Tensor Representation: Sines and Cosines.

136. Suppose that for $1_{b+\frac{n\pi}{2048}}$ $(n < 1)$, on a growth with constant unit tensor from 1_b to $1_{b+\frac{\pi}{2048}}$, we substitute a number on the uniform growth from 1_b to $1_{b+\frac{\pi}{2048}}$, and the same part of the way on that growth that $1_{b+\frac{n\pi}{2048}}$ is on the varying growth. What is the error in tensor and sort?

If $1_b = p + iq$ and $1_{b+\frac{\pi}{2048}} = p' + iq'$, while $m + n = 1$, the number on the uniform growth is $mp + np' + i(mq + nq')$. Of this the tensor squared is

$$\begin{aligned} m^2 + n^2 + 2mn(pp' + qq') &= (m+n)^2 - 2mn(1 - pp' - qq') \\ &= 1 - mn(\overline{p' - p}^2 + \overline{q' - q}^2). \end{aligned}$$

Here $\sqrt{(p'-p)^2+(q'-q)^2}$ does not differ (§ 126) 0.0000000004 from $\frac{\pi}{2048}$. For each we will for brevity write h. Thus the square of the tensor of the substituted number is $1-mnh^2$. Now mn is never larger than $\frac{1}{4}$, its value when $m=n=\frac{1}{2}$. Any other value could, in fact, be written $(\frac{1}{2}+k)(\frac{1}{2}-k)=\frac{1}{4}-k^2$. Consequently $mnh^2 \overline{\overline{<}} \frac{h^2}{4}$, and the error in tensor,

$$1-\sqrt{1-mnh^2}, < \frac{h^2}{4} < 0.0000001.$$

Let the substituted number grow, keeping the sort constant, until the tensor is unity. Suppose that the number then becomes 1_{b+t}. The error in sort is $t-nh$. We have

$$1_{b+t} = \frac{mp+np'}{\sqrt{1-mnh^2}} + i\,\frac{mq+nq'}{\sqrt{1-mnh^2}} \quad \text{and} \quad t = T(1_{b+t}-1_b).$$

$$\text{Hence } t^2 = \left(\frac{mp+np'}{\sqrt{1-mnh^2}}-p\right)^2 + \left(\frac{mq+nq'}{\sqrt{1-mnh^2}}-q\right)^2$$

$$= 2 - \frac{2-nh^2}{\sqrt{1-mnh^2}}.$$

Since $\frac{2-nh^2}{\sqrt{1-mnh^2}} > 2-nh^2$, $k^2 > nh^2$ and *a fortiori* $k^2 > n^2h^2$.

On the other hand, $2 - \frac{2-nh^2}{\sqrt{1-mnh^2}} < \frac{n^2h^2}{1-mnh^2}$. For the truth of this last involves and is involved in

$$\left(2-\frac{n^2h^2}{1-mnh^2}\right)^2 < \frac{(2-nh^2)^2}{1-mnh^2};$$

which after expansion and reduction becomes $n^2 < 1-mnh^2$, and is true because $n^2 < 1-mn < 1-mnh^2$.

Therefore $\dfrac{nh}{\sqrt{1 - mnh^2}} > t > nh$. But because $\sqrt{1 - mnh^2}$ does not differ mnh^2 from 1, $\dfrac{nh}{\sqrt{1 - mnh^2}}$ does not differ $mnh^2 \times nh = mn^2h^3$ from nh, and the error in sort is not as much as 0.0000000001.

Had we the numbers $p + iq$ for intervals in the value of b less than $\dfrac{\pi}{2048}$, the above method of approximation to intermediate $p + iq$ numbers would give still better results.

137. The numbers p and q, depending on b, are called respectively the *cosine* and the *sine* of b, so that

$$\cos b + i \sin b = 1_b.$$

The number b is usually given for the intervals $\pi \div 180$, called *degrees*, $\pi \div 180 \div 60$, called *minutes*, and $\pi \div 180 \div 60 \div 60$, called *seconds*. Thus:

$$\text{one degree} = 1^\circ = 0.0174533 \ldots,$$
$$\text{one minute} = 1' = 0.0002909 \ldots,$$
$$\text{one second} = 1'' = 0.0000048 \ldots,$$
$$\text{while unity} = 57^\circ\ 17'\ 44''.8.$$

Comparing § 135, write in terms of the sines and cosines of b and b', $\cos\left(b + \frac{\pi}{2}\right)$, $\sin\left(b + \frac{\pi}{2}\right)$, $\cos(b + \pi)$, $\sin(b + \pi)$, $\cos(b \pm b')$, $\sin(b \pm b')$, $\cos \frac{1}{2}b$, $\sin \frac{1}{2}b$, $\cos 2b$, $\sin 2b$, $\sin 3b$, $\sin\left(\frac{\pi}{2} - b\right)$, $\cos(-b)$.

Prove that $(\cos b + i \sin b)^m = \cos mb + i \sin mb$,

$$\cos b = \tfrac{1}{2}(e^{ib} + e^{-ib}), \ \sin b = \frac{i}{2}(e^{-ib} - e^{ib}).$$

Show that a table giving the sine and cosine of b for all values of b from zero to $\dfrac{\pi}{4}$ is sufficient for getting the sines and cosines of any value of b.

With such a table let the student find sin 3, sin 1.15, cos $\frac{3}{4}$; numbers whose sines are 2, $4\frac{1}{2}$, -3, -13.

He can roughly verify his values by plotting on the Argand diagram the values of 1_b, given or obtained, together with the corresponding $p+iq$ numbers.

138. Cast a glance back over the route by which we have come.

We started with familiar ideas of counting, and followed them by the scarcely less familiar ideas of addition, subtraction, multiplication, division, involution, and evolution; adding to these, for the sake of completeness, the process of taking a logarithm.

To make subtraction always possible, negative numbers were introduced and defined, and the necessary extensions of the algebraic processes carried out.

In the same way, fractions made division always possible; while for root extraction and taking of logarithms, incommensurables and double numbers were needed.

In all of our extensions of meanings it has been found possible to adhere to these rules:

The extended meaning has included the unextended as a special case.

The relation of process and inverse has been maintained.

The commutative, associative, distributive, and index laws that held with the first simplest numbers have been made to hold throughout.

Finally, by our extensions, we have made the three primary operations and their four inverses always possible, i.e. always resulting in a number belonging to our scheme.

Notice now whither further progress may lead us.

In the reasoning first employed for the handling of incommensurables and further developed in the treatment of growths and rates we have the germ of that marvellous invention of Leibnitz and Newton, the Infinitesimal Calculus.

The graphic representation leads to the Analytic Geometry of Des Cartes.

The theory of sines and cosines with its geometric applica-

tions is Trigonometry and leads, by the introduction of new numbers and conceptions, to the Function Theory.

The theory of double numbers is simple and restricted, and but a faint suggestion of what is to be found in the beautiful developments proposed by Hamilton and Grassmann, the Peirces and Sylvester. These are a few only of the lines of thought open to the student.

Yet, in whatsoever direction investigation may carry him, he will find his work essentially the same in character. Definitions and conventions and their logical consequences and relations make up the whole of it. These relations form the universe wherein the Mathematician lives; a universe, to be sure, of his own construction, a product of his brain, but none the less real and substantial to him. Here he observes and compares and experiments; here he reasons out connections, discovers causes, and foretells results.